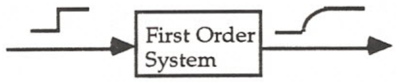
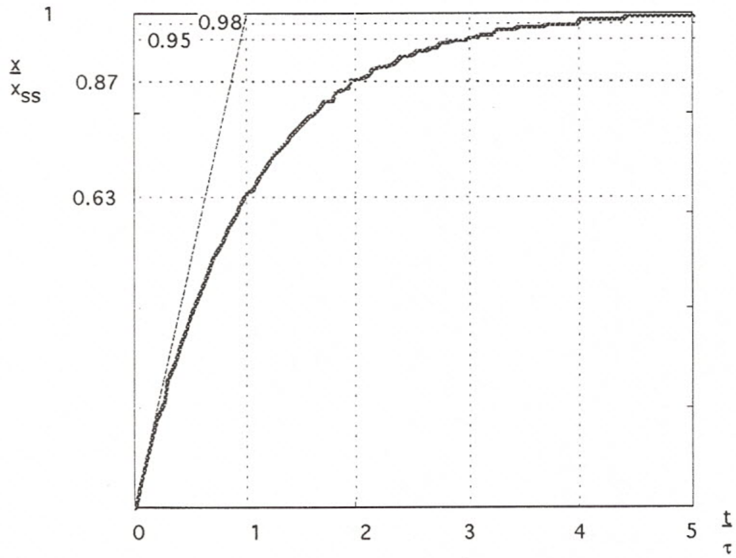


Step Response $u(t)$ = ***unit*** step = 1 t ≥ 0

$\dot{x} = ax + bu \qquad x(0) = 0$

$$x(t) = -\frac{b}{a}(1 - e^{at}) = b\tau(1 - e^{-t/\tau})$$

Steady-state value $x_{ss} = -\frac{b}{a} = b\tau$



Initial Slope:

@t=0 $x(0) = 0$; $\frac{dx}{dt} = -\frac{b}{a} = b\tau$

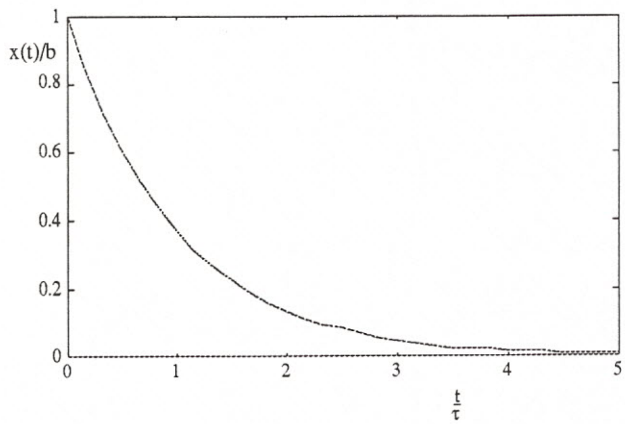
Impulse Response:

Since the unit impulse $\delta(t) = \frac{d}{dt}(\text{step})$

then $x(t)_{impulse} = \frac{d}{dt}(x(t)_{step})$;

$$= \frac{d}{dt}(b\tau(1-e^{-t/\tau}) = b\, e^{-t/\tau}$$

which looks the *same* as the homogenous, initial condition response, i.,e. a simple decaying exponential.



Repeated Roots for Second Order Systems:

For the differential equation: $\ddot{y}+2\zeta\omega_n\dot{y}+\omega_n^{\ 2}y=Ku(t)$, when $\zeta = 1.0$, the system is said to be ***critically damped*** and solution of the characteristic equation:
$s\ +2\zeta\omega_n s+\omega_n^{\ 2}=0$ leads to two identical roots $s_1=s_2=-\zeta\omega_n=\omega_n$.

For this special case the homogenous solutions is given by:

$y_h(t) = A_1 e^{s_1 t} + A_2\, t\, e^{s_1 t}$

For the initial conditions : $y(0) = y_0$ and $\dot{y}(0)=0$, the solution for A_1 and A_2 is:

$A_1 = y_0$
$A_2 = -\ s_1 A_1 \ = -s_1 y_0$

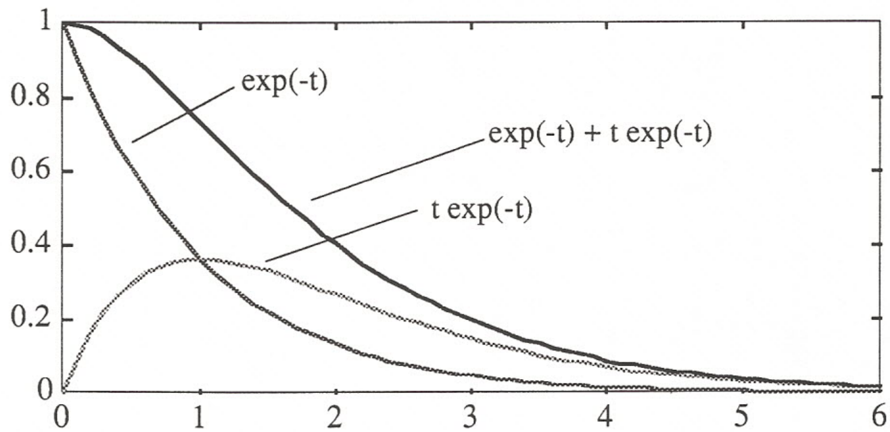
So the Homogeneous solution is given by:

$y_h(t) = y_0(e^{s_1 t} - s_1 t\, e^{s_1 t})$

If we now substitute $s_1 = -\omega_n$, the solution becomes:

$y_h(t) = y_0(e^{-\omega_n t} + \omega_n t e^{-\omega_n t})$

The two terms in this equation are shown below for the case of $\omega_n = 1$, and $y_0 = 1.0$:



Homogenous Response for ζ=1, ω_n = 1 and y(0) = 1.0

Note that the final curve for y(t) ***does not go below zero***, but instead asymptotically approaches the final value of 0 from above.

Non-Dimensional Step Response for Second Order Systems

The step response of an underdamped second order system defined by the equation

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = Ku(t) \ ,$$

when $y(0) = 0$ and $\dot{y}(0) = 0$, and

y_{SS} is the final value of y as t->∞ is given by:

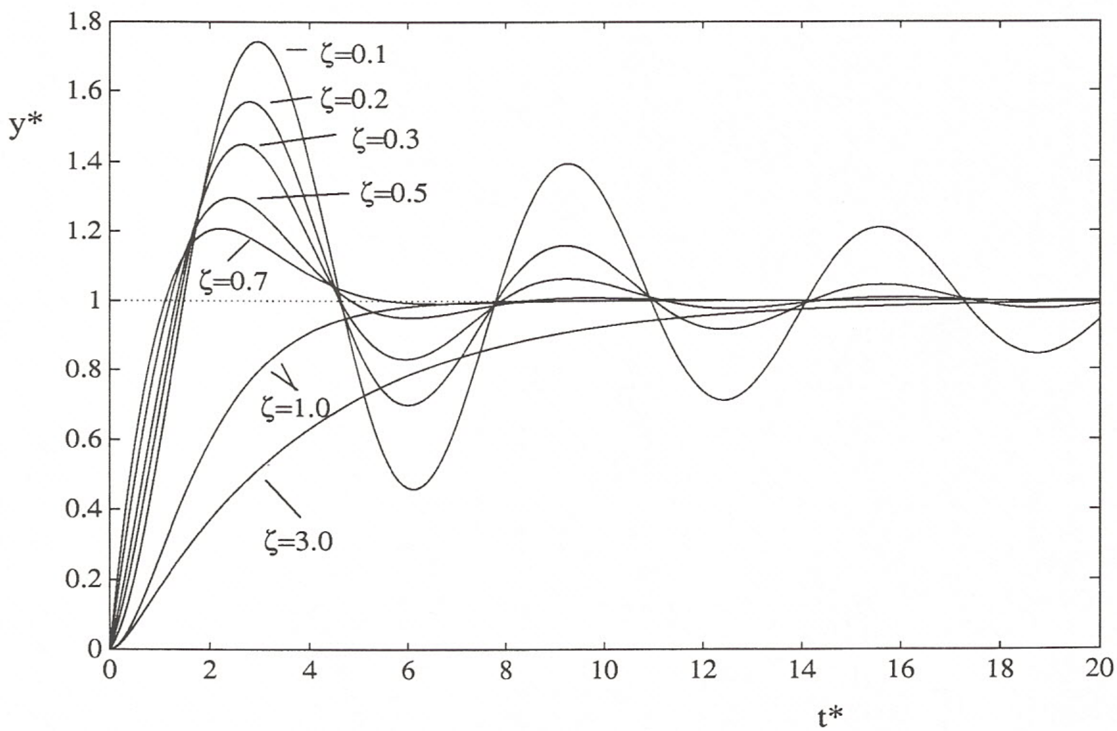
$$y(t) = y_{ss}\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\cos(\omega_d t + \psi)\right] \qquad \text{for } 0 < \zeta < 1 \text{ and } \quad \psi = \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)$$

Since ω_n always appears as the product $\omega_n t$; we can plot this product ass the independent variable. This implies that ω_n serves as a *time scalar* for the response. This also shows that the damping ratio ζ serves to scale the *shape* of the response, since it now determines the response envelope $exp(-\zeta\ \omega_n t)$.

We can also non-dimnesionalize the output by dividing through by y_{SS} thereby creating the equation:

$$y^* = \left[1 - \frac{e^{-\zeta t^*}}{\sqrt{1-\zeta^2}}\cos(\sqrt{1-\zeta^2}\,t^* + \psi)\right] \quad \text{where } t^* = \omega_n t \text{ and } y^* = \frac{y(t)}{y_{ss}}$$

This equation can be plotted for various values of ζ:



Step Response Specifications:

Various measures are used to define the step response of a dynamic system. These include:

M_p = percent overshoot

$t_s|_{2\%}$ = time to stay within ± 2% of the final value ($t_s|_{5\%}$ is also commonly used)

$t_r|_{100\%}$ = rise time= time to reach 100% of final value **the first time.**

Since a critically or overdamped system will never reach 100% of the final value, we also define other rise times, such as:

$t_r|_{90\%}$ = time to reach 90% of final value

t_p = the peak time = time of maximum overshoot.

These measures are applicable to the step response of any system (regardless of its order), but they have specific analytical values for first and second order systems.