The Exponentially Weighted Moving Average

EWMA good for understanding and control.

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See pp 208-9 on control

The Shewhart and CUSUM control chart techniques have found wide application in the manufacturing industries. However, workpiece quality has also been greatly enhanced by rapid and precise individual item measurements and by improvements in automatic dynamic machine control. One consequence is a growing similarity in the control problems faced by the workpiece quality control engineer and his compatriot in the continuous process industries. The purpose of this paper is to exposit a control chart technique that may be of value to both manufacturing and continuous process quality control engineers: the exponentially weighted moving average (EWMA) control chart. The EWMA has its origins in the early work of econometricians, and although its use in quality control has been recognized, it remains a largely neglected tool. The EWMA chart is easy to plot, easy to interpret, and its control limits are easy to obtain. Further, the EWMA leads naturally to an empirical dynamic control equation.

Introduction

THE exponentially weighted moving average (EWMA) has been around a long time. The EWMA in quality control charting was first exposited by Roberts (1959) in a landmark paper in which the author compared the average run lengths of the EWMA chart (there called "the geometric moving average chart") to the Shewhart chart and other simple moving average schemes. The EWMA also found early application in economics (Muth [1960]), and in inventory control and forecasting (Brown [1962]). The industrial uses of the EWMA were discussed by Freund (1962), and modifications by Stewart (1970), and Wortham and Henrich (1972). The EWMA has, however, seldom been employed by quality control engineers. This is doubly unfortunate since the EWMA can be viewed as a compromise between the Shewhart and CUSUM charting procedures, and perhaps more importantly, as a method for establishing real-time dynamic control of industrial processes. We begin with a brief description of the associated Shewhart and CUSUM charting procedures.

In all that follows, observations y_t ($t = 1, 2, \cdots, T$) are assumed to be sequentially recorded and the observations, or some function of the observations, plotted for the purpose of controlling a manufacturing process. In each case the plotted point is compared to control limits, and if found to be beyond these critical boundaries, declared a signal that the observed process requires attention. In this paper all descriptions and comparisons of charting techniques will assume the observations to be normally distributed. Thus $y_t = \eta + \epsilon_t$ where η is the mean of the process and the errors ϵ_t are normally and independently distributed with zero mean and constant variance.

The Shewhart Control Chart

In a Shewhart control chart upper and lower control limits for the plotted points are established around the process mean η at positions $\eta \pm 3\sigma$, the "three-sigma" limits where σ is the standard error of the points plotted. In a process under ideal control the mean η will equal a fixed target value τ required by the process specifications. A signal that the observed process requires attention is given whenever the most recent plotted point falls outside the control limits. This method of process control charting is called a Shewhart chart to honor its originator, Dr. W. A. Shewhart (1931, 1938).

In practice the Shewhart control chart is established empirically. Usually, repeated random homogeneous samples of four or five observations each are gathered from the process, and, based upon the sample averages $\bar{y}$ and ranges R (or estimates s of the standard deviation), empirical control charts are established with the center line of the chart the grand average $\bar{\bar{y}}$. Many approaches for establishing the three-sigma upper and lower control limits for the Shewhart control chart are in use, each dependent upon knowledge about the mean, variance and target values and each with its own set of constants. See ANSI/ASQC A1 (1978, Table 1), Grant (1952), Hansen (1963), Duncan (1974), and Grant and Leavenworth (1980).

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The standard Shewhart chart with its $\pm 3\sigma$ limits only rarely provides a signal when the current process is at its mean level, thus keeping the Type I error, or "α risk," small. Freund's (1957) acceptance control charts provide control limits that simultaneously consider both the Type I and Type II errors (i.e., both α and β).

The CUSUM Chart

An alternative method for plotting sequentially recorded observations y_t is to plot their cumulative sum, $\sum_{t=1}^{T} y_t$, against time t. Or, rather than the cumulative sum of the observations, one can use the cumulative sum of the deviations

$$d_t = y_t - C \tag{1}$$

where C is some convenient constant, usually the target value τ. Thus a CUSUM chart is the quantity

$$S_T = \sum_{t=1}^{T} (y_t - C) = \sum_{t=1}^{T} d_t \tag{2}$$

plotted against T.

The CUSUM control chart was first introduced in England by Page (1954). Other important early contributors are Barnard (1959), Ewan and Kemp (1960) and Johnson and Leone (1962). Ewan (1963) provides an excellent expository article on the CUSUM, and a small text elucidating CUSUM procedures was authored by Woodward and Goldsmith (1964).

If the mean η of the observed values y_t equals the target value, τ, then $S_T = \sum_{t=1}^{T} d_t$ will plot as a random walk, that is, S_T will wander randomly about zero. However, should the mean of the process differ by a slight amount δ from the target value, then the expected value of S_T will add δ with each observation. The plot of S_T thus increases or decreases depending on the sign of δ. Therefore, when plotting a CUSUM chart the analyst watches for a change in the slope of the plot of S_T as an indication of a shift in mean away from target.

Surprisingly, even when the mean is on target, the CUSUM S_T can wander remarkably far and give the appearance of a change in mean. For statistical process control purposes the CUSUM chart is often accompanied by a "V" mask. The mask is in the shape of a V placed sidewise (>) with its vertex placed a fixed distance from the last plotted CUSUM point. As long as the entire historical track of S_T lies within the branches of the V, no signal is provided. A signal that the process needs attention occurs when any single point of the S_T plot falls outside the branches of the V.

Unlike the Shewhart chart, the control limit parameters (the lead distance and interior angle of the V mask) take into account both the α and β risks along with the size of the shift δ felt to be economically important. Detailed instructions on the construction of the V mask are given in Johnson and Leone (1977).

Many variations of the CUSUM are possible. If in equation (2) one uses the quantity $C = \tau \pm \delta/2$, the V mask converts into two separate pairs of parallel control limits similar in appearance to the Shewhart chart, one pair of control limits for $C = \tau + \delta/2$, and a second for $C = \tau - \delta/2$. A single pair of these control limits can be used as a one-sided CUSUM chart. In these instances only those values of S_T with the same sign as δ are plotted. For example, should δ be positive and the CUSUM S_T go negative because of an observed series of negative recordings of $d_t = y_t - (\tau + \delta/2)$, then S_T is simply recorded as equal to zero. A one-sided CUSUM chart thus occasionally "ignores" observed values of d_t having a sign opposite to δ. The plot of S_T restarts with the first d_t with the same sign as δ. Recently Lucas and Crosier (1982) have proposed important modifications to the CUSUM chart that provide for early detection of process shifts.

Weighting the Historical Data

One key to understanding the differences between the Shewhart, CUSUM, and EWMA control charts rests in knowing how each charting technique uses the data generated by the production process. For the classical Shewhart chart (where by "classical" we mean for the moment to ignore the use of runs) the signal that the process may be out of control is the appearance of a single point falling beyond the 3σ limits. In terms of a *weighting function*, the signal thus depends entirely on the last plotted point; that is, the weight given the last plotted point is $w_t = 1$, and the weights given to all previous values are $w_{t-k} = 0$ for $k \geq 1$. For the ordinary CUSUM chart the out of control signal depends upon the sum $S_T = \sum_{t=1}^{T} d_t$. Of course, in computing a sum each entry has equal weight, and thus the most recent d_t is equal in influence to the most ancient, and $w_t = 1/n$ for all t.

The weighting functions for the classical Shewhart chart and for the ordinary CUSUM chart are displayed in Figures 1a and 1b. The reader will note that for the purpose of providing signals the classical Shewhart chart has no memory. That is, it ignores the immediate history. The ordinary CUSUM chart, however, has an elephant-like memory, giving equal attention to the

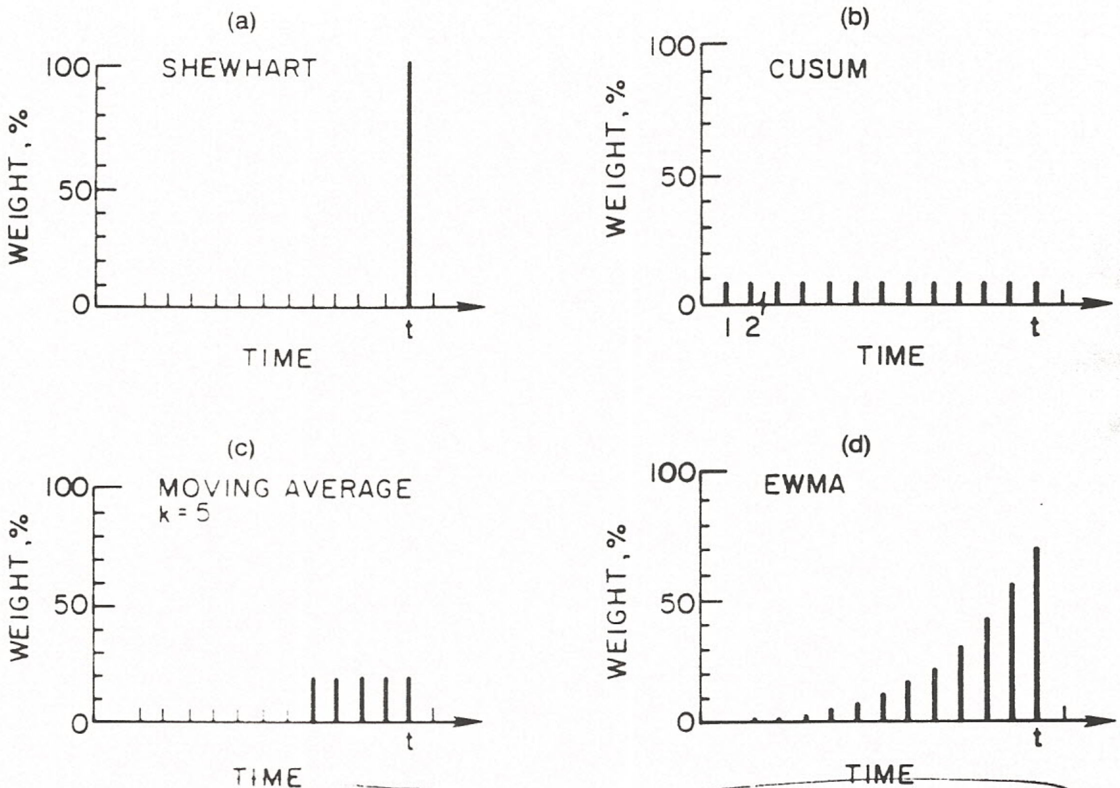

FIGURE 1. Data Weighting for the Shewhart, CUSUM, Moving Average and EWMA Charts.

most ancient datum as well as the most recent. Of course, the Shewhart chart with runs does use the historical data, and various forms of the CUSUM chart will ignore portions of the history. In both instances the data weighting functions will therefore differ from those displayed. Our purpose here is to illustrate in its simplest form how the charts weight the data.

The Moving Average

The desire to employ historical data more resourcefully has occasionally led to the use of the moving average. A plot of a moving average of $k = 5$ observations simply displays the average of the five most recent observations. Each new entering observation forces the oldest in the group out of the computation. The weighting function for a moving average of $k = 5$ is displayed in Figure 1c. One may question the reasonableness of giving equal importance to the most recent k observations and zero importance to all previous. The moving average smooths a time series. Further, as illustrated in Nelson (1983), the moving average can produce cyclic and trend-like plots even when the original data are themselves independent random events with a fixed mean. This characteristic lessens its usefulness as a control mechanism.

The EWMA

The exponentially weighted moving average (EWMA) is a statistic with the characteristic that it gives less and less weight to data as they get older and older. A plotted point on an EWMA chart can be given a long memory, thus providing a chart similar to the ordinary CUSUM chart, or it can be given a short memory and provide a chart analogous to a Shewhart chart. An example of a weighting function for a EWMA is displayed in Figure 1d.

The EWMA is very easily plotted and may be graphed simultaneously with the data appearing on a Shewhart chart. The EWMA is best plotted one time position ahead of the most recent observation. Later discussion will show the EWMA may be viewed as the forecast for the next observation, but that need not bother us here. Our immediate purpose is only to plot the statistic. The EWMA equals the present predicted value plus λ times the present observed error of prediction. Thus,

$$\text{EWMA} = \hat{y}_{t+1} = \hat{y}_t + \lambda e_t \tag{3}$$

$$= \hat{y}_t + \lambda(y_t - \hat{y}_t) \tag{4}$$

where

$\hat{y}_{t+1}$ = predicted value at time $t+1$ (the new EWMA)
y_t = observed value at time t
$\hat{y}_t$ = predicted value at time t (the old EWMA)
$e_t = y_t - \hat{y}_t$ = observed error at time t
and λ is a constant $(0 < \lambda < 1)$ that determines the depth of memory of the EWMA.

Equation (4) can be written as

$$\hat{y}_{t+1} = \lambda y_t + (1-\lambda)\hat{y}_t. \qquad (5)$$

Plotting the EWMA

To plot the EWMA, let the symbol * stand for predicted values and ● for observed values. We now propose to plot simultaneously the observations y_t and the EWMA $\hat{y}_{t+1}$ on the same graph. Data for the example are given in Table 1. Let $\lambda = 0.5$ and suppose at $t = 0$ a process thought to be under control has a target value $\tau = 50$. To initiate the EWMA, set the initial predicted value $\hat{y}_1$ equal to the target value and plot it at position $t = 1$. The first observed value $y_1 = 52$ is now also plotted at position $t = 1$ as shown in Figure 2a. The observed error e_1 equals 2.0. The next predicted value $\hat{y}_2$ is obtained using equation (3). Thus $\hat{y}_2 = 50 + 0.5(52 - 50) = 51$ as illustrated in Figure 2b. When the second observed value $y_2 = 47$ is obtained, Figure 2c can be constructed. The observed value y_2 is first plotted and then the new predicted value $\hat{y}_3 = \hat{y}_2 + \lambda e_2 = 51 + 0.5(47 - 51) = 49$. The fourth predicted value, displayed in Figure 2d, is obtained once the third observed value $y_3 = 53$ is in hand. Thus $\hat{y}_4 = \hat{y}_3 + 0.5(y_3 - \hat{y}_3) = 51$, and so on. The predicted values, the $\hat{y}_{t+1}$, are the EWMA.

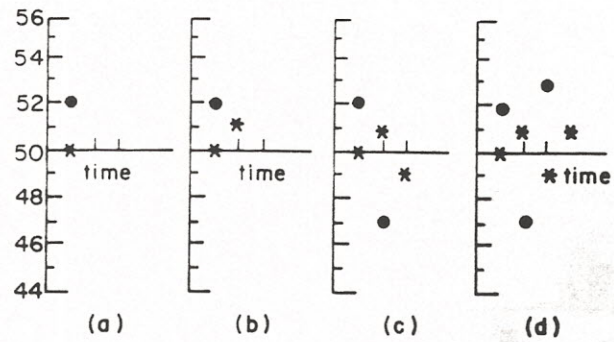

FIGURE 2. The EWMA, Indicated by the *, is Computed at Time t but Plotted at Time Position $t + 1$. Observations are indicated by ●.

The current EWMA contains all the historical information and there is no need to keep records of the previous observations or their weights. The new EWMA, $\hat{y}_{t+1}$, is simply updated by adding λe_t to the old EWMA, $\hat{y}_t$. After a little practice, plotting the EWMA is almost as easy as plotting the successive observations.

TABLE 1. EWMA Forecast Computations Using Various Values of λ

		$\lambda = 0.5$		$\lambda = 0.8$		$\lambda = 0.2$	
t	obs	EWMA	e_t	EWMA	e_t	EWMA	e_t
1	52.0	(50.00)	2.00	(50.00)	2.00	(50.00)	2.00
2	47.0	51.00	−4.00	51.60	−4.60	50.40	−3.40
3	53.0	49.00	4.00	47.92	5.08	49.72	3.28
4	49.3	51.00	−1.70	51.98	−2.68	50.38	−1.08
5	50.1	50.15	−0.05	49.84	0.26	50.16	−0.06
6	47.0	50.13	−3.13	50.05	−3.05	50.15	−3.15
7	51.0	48.56	2.44	47.61	3.39	49.52	1.48
8	50.1	49.78	0.32	50.32	−0.22	49.82	0.28
9	51.2	49.94	1.26	50.14	1.06	49.87	1.33
10	50.5	50.57	−0.07	50.99	−0.49	50.14	0.36
11	49.6	50.54	−0.94	50.60	−1.00	50.21	−0.61
12	47.6	50.07	−2.47	49.80	−2.20	50.09	−2.49
13	49.9	48.83	1.07	48.04	1.86	49.59	0.31
14	51.3	49.37	1.93	49.53	1.77	49.65	1.65
15	47.8	50.33	−2.53	50.95	−3.15	49.98	−2.18
16	51.2	49.07	2.13	48.43	2.77	49.55	1.65
17	52.6	50.13	2.47	50.65	1.95	49.88	2.72
18	52.4	51.37	1.03	52.21	0.19	50.42	1.98
19	53.6	51.88	1.72	52.36	1.24	50.82	2.78
20	52.1	52.74	−0.64	53.35	−1.25	51.37	0.73
21		52.42		52.35		51.52	

The Weighting Constant λ

As shown in the Appendix, the EWMA can be written as

$$y_{t+1} = \sum_{i=0}^{t} w_i y_i \tag{6}$$

where the w_i are weights and

$$w_i = \lambda(1-\lambda)^{t-i} \tag{7}$$

The sum of the weights $\sum_{i=0}^{t} w_i = 1$. The constant λ determines the "memory" of the EWMA statistic. That is, λ determines the rate of decay of the weights and hence the amount of information secured from the historical data. Note now that as $\lambda \rightarrow 1$, $w_1 \rightarrow 1$ and $\hat{y}_{t+1}$ practically equals the most recent observation y_t. When the process is under control and $\lambda = 1$, points plotted on the classical Shewhart chart and those on an EWMA chart are therefore almost equal in their ability to detect signals of departures from assumptions. As $\lambda \rightarrow 0$, the most recent observation has small weight, and previous observations near equal (though lower) weights. Thus, as $\lambda \rightarrow 0$ the EWMA takes on the appearance of the CUSUM. The EWMA control chart for values of $0 < \lambda < 1$ stands between the Shewhart and CUSUM control charts in its use of the historical data.

The choice of λ can be left to the judgment of the quality control analyst. Experience with econometric data suggests values of $\lambda = 0.2 \pm 0.1$. The smaller the value of λ the greater the influence of the historical data. For example, using equation (7) with $\lambda = 0.3$, the most recent y_t has weight 0.3 while y_{t-10} has weight $\lambda(1-\lambda)^{10} = 0.0085$; that is, only $w_{t-10}/w_t = 0.028$ or approximately 3% the weight of the most recent observation. However, with $\lambda = 0.1$ observation y_{t-10} has weight $w_{t-10} = 0.0349$ or approximately 35% the weight of the most recent observation.

The parameter λ can be estimated using an iterative least squares procedure. To illustrate, suppose the observations $y_1, y_2, \cdots, y_{20}$ listed in Table 1 had been gathered from a process with target value $\tau = 50$ *before* λ had been chosen. The analyst, by considering the data as new data arriving sequentially, could for different values of λ compute the corresponding sequential set of predicted values $\hat{y}$ based on the EWMA as illustrated in Table 1. For $\lambda = 0.5$ the sum of squares of the errors associated with the predicted values in Table 1 is $\sum_{t=1}^{20} e_t^2 = 89.66$, for $\lambda = 0.8$ the error sum of squares is $\sum_{t=1}^{20} e_t^2 = 117.39$ and for $\lambda = 0.2$, $\sum_{t=1}^{20} e_t^2 = 78.02$.

Based upon this very limited evidence the value $\lambda = 0.2$ is preferred since it provides the smallest error sum of squares. The 'best' estimate, the least squares estimate, is $\lambda = 0.112$. In practice, at least fifty observations should be employed in such estimation procedures.

EWMA Control Limits

As shown in the Appendix, the variance of the EWMA is

$$\text{Var(EWMA)} = [\lambda/(2-\lambda)]\sigma^2$$

and thus

$$\sigma_{\text{EWMA}} = [\lambda/(2-\lambda)]^{0.5}\sigma.$$

An estimate of σ^2 can be obtained from the minimum error sum of squares $\sum e_t^2$ obtained while estimating λ. That is,

$$\hat{\sigma}^2 = \sum_{t=1}^{T} e_t^2/(T-1).$$

When the Shewhart and EWMA charts are constructed and plotted simultaneously σ_{EWMA} can be estimated from the information used in establishing the three-sigma Shewhart control limits. That is,

$$\hat{\sigma}_{\text{EWMA}} = [\lambda/(2-\lambda)]^{0.5}\hat{\sigma}_{\text{Shewhart}}.$$

The three-sigma control limits for the EWMA are

$$\tau \pm 3[\hat{\sigma}_{\text{EWMA}}].$$

For this example, with the target value $\tau = 50$ and $\lambda = 0.5$ the three-sigma limits for the EWMA are

$$50 \pm 1.73\hat{\sigma}_{\text{Shewhart}}.$$

The Shewhart and the EWMA control limits may be conveniently placed on the same chart, and both y_t and the EWMA co-plotted as indicated in Figure 3. Using $\hat{\sigma}_{\text{Shewhart}} = 1.5$ the three-sigma Shewhart control limits for the plotted Shewhart points (usually averages) are given by 50 ± 4.5, and the three-sigma control limits for the plotted EWMA forecasts (constructed from the Shewhart points) by 50 ± 2.6. In either case a "signal" is produced whenever the last plotted point falls beyond its appropriate control limits. In this example, at time position $t = 19$ the EWMA (the forecast) falls beyond the upper control limit for the EWMA, a signal that the process needs immediate attention.

In practice the classical Shewhart chart (only the last plotted point falling outside the 3σ limits providing a signal) is aided by the use of runs. For example, two out of three successive points falling within the region $\eta \pm 2\sigma$ and $\eta \pm 3\sigma$, or six points in sequence above the chart's center line, are frequently taken to be signals that the process is no longer under control, see Nelson (1984). Employing runs provides an informal use of

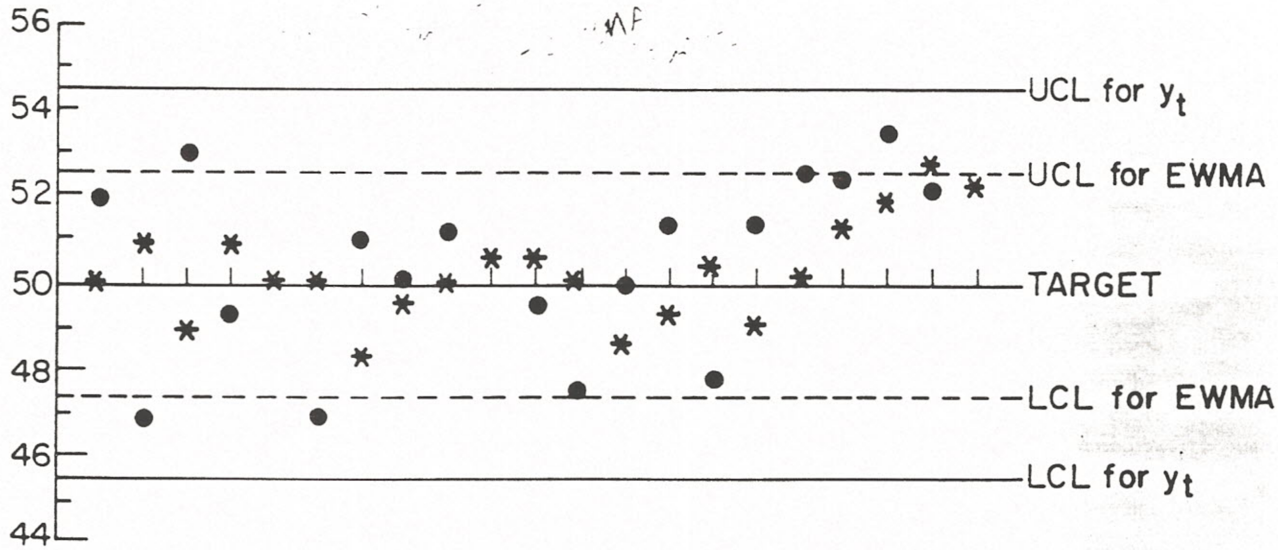

FIGURE 3. Observations y (●) and the EWMA (*) Co-plotted With Their Upper and Lower Control Limits.

the recent history and, in the hands of an experienced analyst, can make the Shewhart chart take on the aspects of the EWMA chart. However, the EWMA chart provides a regular and formal use of the historical data. Runs and all other data configurations are encompassed in the EWMA forecast. Further, once the EWMA has been computed it contains all the information provided by the historical record. There is no need to save or, as in the case of the modified CUSUM, to give occasional zero weights to recorded observations.

Process Control

The EWMA can be used as a dynamic process control tool. When the process mean η is on the target (i.e., $\eta = \tau$) all three charting procedures, the Shewhart, CUSUM and EWMA, are roughly equivalent in their ability to *monitor* departures from target. However, the EWMA provides a forecast of where the process will be in the next instance of time. It thus provides a mechanism for *dynamic process control.*

To control a process it is convenient to forecast where the process will be in the next instance of time. Then, if the forecast shows a future deviation from target that is too large, some electro-mechanical control system or process operator can take remedial action to compel the forecast to equal the target. In modern manufacturing, and particularly where an observation is recorded on every piece manufactured or assembled, a forecast based on the unfolding historical record can be used to initiate a feedback control loop to adjust the process.

Of course, if an operator is part of a feedback control loop he must know what corrective action to perform, and care must be taken to avoid inflating the variability of the process by making changes too often. But control engineers long ago learned how to close the feedback loop linking forecast and adjustment to target. The same information feedback loop exists in many situations which only the operator can control. The EWMA chart not only provides the operator with a forecast, but also with control limits to inform when the forecast is statistically significantly distant from the target. Thus, when an EWMA signal is obtained, appropriate corrective action based on the size of the forecast can often be devised.

The Empirical Control Equation

The EWMA can be modified to enhance its ability to forecast. In situations where the process mean steadily trends away from target the EWMA can be improved by adding a second term to the EWMA prediction equation. That is,

$$\text{Modified EWMA} = \hat{y}_{t+1} = \hat{y}_t + \lambda_1 e_t + \lambda_2 \sum e_t$$

where λ_1 and λ_2 are constants that weight the error at time t and the sum of the errors accumulated to time t. The coefficients λ_1 and λ_2 can be estimated from historical data by an iterative least-squares procedure similar to that illustrated earlier for the estimation of λ in equation (3).

A third term can be added to the EWMA prediction equation to give the *empirical control equation*

$$\hat{y}_{t+1} = \hat{y}_t + \lambda_1 e_t + \lambda_2 \sum e_t + \lambda_3 \nabla e_t$$

where the symbol ∇ means the first difference of the errors e_t; that is $\nabla e_t = e_t - e_{t-1}$. Note now that the forecast $\hat{y}_{t+1}$ equals the present predicted value (zero if the process has been adjusted to the target) plus three quantities: one *proportional* to e_t, the second a function of the *sum* of the e_t, and the third a function of the first *difference* of the e_t. These terms are sometimes called the "proportional", "integral", and "differential" terms in the process control engineer's basic proportional, integral, differential ("PID") control equation. The parameters λ_1, λ_2 and λ_3 weight the historical data to give the best forecast.

The EWMA may thus be viewed as more than an alternative to either the Shewhart or the CUSUM control charts. The EWMA may also be viewed as a *dynamic control* mechanism to help keep a process mean on target whenever discrete data on manufactured items are sequentially available. An extensive literature exists describing discrete data process control, two important statistical references being Box and Jenkins (1976) and more recently, Pandit and Wu (1983). In this literature one finds that the EWMA is an important member of a large class of time series models identified by Box and Jenkins as ARIMA models (Autoregressive, Integrated, Moving Average models).

Conclusion

The technology of industrial quality control is a continuum, beginning with simple time plots of data and proceeding to control chart methodologies of increasing complexity. The Shewhart, CUSUM and EWMA control charts all represent stages in this continuum. The EWMA chart may be viewed as an alternative, or used in addition to, the more familiar Shewhart and CUSUM charts. But the EWMA chart also provides the quality control engineer with the opportunity to begin to consider real-time dynamic control of processes using discrete data. Most applications of control charting techniques require that the operator leave the process alone. If desired, the EWMA can be employed to make the operator part of the feedback control loop.

Acknowledgments

The author has discussed the EWMA with many associates, most particularly G. E. P. Box. He is also indebted to Jack Wood and Donald Gardiner for their reading of an earlier manuscript, to the students in many industrial short courses, and most particularly to the referees and the Editor for valuable commentary.

Appendix

Substituting $\hat{y}_t = \lambda y_{t-1} + (1-\lambda)\hat{y}_{t-1}$ into equation (5) gives for the EWMA

$$\hat{y}_{t+1} = \lambda y_t + (1-\lambda)[\lambda y_{t-1} + (1-\lambda)\hat{y}_{t-1}]$$
$$= \lambda y_t + \lambda(1-\lambda)y_{t-1} + (1-\lambda)^2\hat{y}_{t-1}$$

but

$$\hat{y}_{t-1} = \lambda y_{t-2} + (1-\lambda)\hat{y}_{t-2}.$$

Thus

$$\hat{y}_{t+1} = \lambda y_t + \lambda(1-\lambda)y_{t-1} + \lambda(1-\lambda)^2 y_{t-2} + (1-\lambda)^3\hat{y}_{t-2}.$$

Employing equation (5) recursively we find that the EWMA is a linear combination of the observations of the form

$$\hat{y}_{t+1} = \text{EWMA} = \sum_{i=0}^{t} w_i y_i$$

where the weights

$$w_i = \lambda(1-\lambda)^{t-i}.$$

Given the errors e_i are independent with constant variance, the variance of the EWMA is

$$\text{V(EWMA)} = (\sum_{i=0}^{t} w_i^2)\sigma^2$$
$$= [\lambda^2 + \lambda^2(1-\lambda)^2 + \lambda^2(1-\lambda)^4 + \cdots]\sigma^2$$
$$= \lambda^2\sigma^2/[1-(1-\lambda)^2] = [\lambda/(2-\lambda)]\sigma^2.$$

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Key Words: *Cumulative Sum Control, Forecasting, Moving Averages, Process Control, Shewhart Control Charts.*