Basic Linear Algebra

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Coordinate Systems

Points are positions in space. When first introduced in Greek geometry, there was no formal method for measuring where a point was located. Points were primitive entities. Much later in history, in the early seventeenth century, Fermat and Descartes introduced the idea of using a linear **coordinate system** to specify point locations, using algebraic equations to describe geometric objects (such as lines), and using algebra to solve geometric problems (such as computing the intersection point of two lines).

A coordinate system has an **origin** as an absolute reference point whose location is given a priori, and **axes** (a set of vectors) that determine the directions in which to make measurements. A coordinate system specifies points in n-dimensional space as a set of numbers $(x_1, x_2,\ldots, x_n)$ and the numeric rule: start at the origin, go distance x_1 in the direction of the first axis, stop, now go distance x_2 in the direction of the second axis, stop, and so on until done with x_n. The final location that one reaches is the point specified. This is much like finding the treasure on a pirate's map; for example, start at the old oak tree as the origin, first go east 10 paces, second go north 20 paces, and third dig down 3 feet to locate the treasure. The ordered set of numbers $(x_1, x_2,\ldots, x_n)$ is called the **coordinate** of the point it specifies, and a coordinate system is said to **span** the set of points that it can specify. The set of all these points is called the **coordinate space**. A specific set of axes spanning a coordinate space is called a **frame of reference** or a **basis** for the space. Clearly, different coordinate systems can span the same coordinate space, just as the pirate may have used any of several origin trees or orientation directions for his treasure map.

Points and Vectors

Basic Definitions

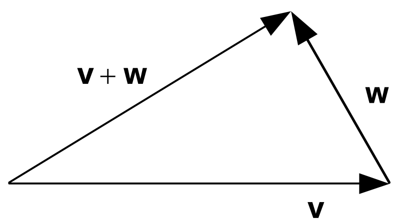
A **scalar** represents magnitude, and is given by a real number: a, b, c,..., x, y, z.

A **point** in n-dimensional space is given by an n-tuple $P = (p_1, p_2, \ldots, p_n)$ where each coordinate p_i is a scalar number. We will write $P = (p_i)$ as a shorthand for this n-tuple. The position of a point is relative to a coordinate system with an origin $\mathbf{0}=(0,0,\ldots,0)$ and unit axes $\mathbf{u}_1 = (1,0,\ldots,0)$, $\mathbf{u}_2 = (0,1,\ldots,0)$, and $\mathbf{u}_n = (0,\ldots,0,1)$. Thus, a 3-dimensional (3D) point is given by a triple $P = (p_1, p_2, p_3)$ whose coordinates are relative to the axes (1,0,0), (0,1,0), and (0,0,1) which are usually referred to as the x, y, and z-axes. Because of this well-known convention, we sometimes write $P = (x, y)$ for 2D points and $P = (x, y, z)$ for 3D points.

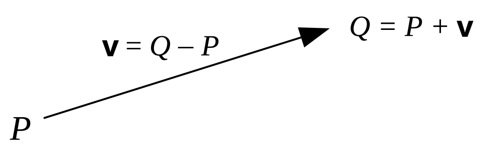
A **vector** represents magnitude and direction in space, and is given by an n-tuple $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ where each coordinate v_i is a scalar. We also write $\mathbf{v} = (v_i)$ as a shorthand for the vector's n-tuple. The vector $\mathbf{v}$ is interpreted to be the magnitude and direction of the line segment going from the origin $\mathbf{0} = (0,0,\ldots,0)$ to the point $(v_1, v_2, \ldots, v_n)$. However, the vector is not this point, which only gives a standard way of visualizing the vector. We can also visualize a vector as a directed line segment from an initial point $P = (p_1, \ldots, p_n) = (p_i)$ to a final point $Q = (q_1, \ldots, q_n) = (q_i)$. Then, the vector from P to Q is given by: $\mathbf{v} = Q - P = (q_1 - p_1, \ldots, q_n - p_n) = (q_i - p_i)$, showing that the difference of any two points is a vector. In particular, vectors do not have a fixed position in space, but can be located at any initial base point P. For example, a traveling vehicle can be said to be going east (direction) at 50 mph (magnitude) no matter where it is located.

Vector Addition

The **sum** of two vectors is given by adding their corresponding coordinates. So, for vectors $\mathbf{v} = (v_i)$ and $\mathbf{w} = (w_i)$, we have: $\mathbf{v} + \mathbf{w} = (v_i + w_i)$. This can be viewed geometrically as:



One can also add a vector $\mathbf{v} = (v_i)$ and a point $P = (p_i)$ by adding their coordinates to get another point $Q = P + \mathbf{v} = (p_i + v_i)$. The resulting point Q is the displacement, or translation, of the point P in the direction and by the magnitude of the vector $\mathbf{v} = P - Q$ as shown by:



Vector addition satisfies the following properties:

- $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ [Association]
- $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$ [Commutation]

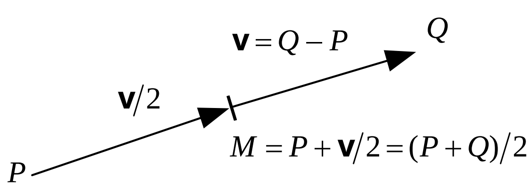
Scalar Multiplication

Multiplication of a vector by a scalar number is given by the formula: $a\mathbf{v} = (av_i)$, multiplying each component of $\mathbf{v} = (v_i)$ by the scalar a. This represents *scaling* the size of a vector by a magnification factor of a. So, for example, $2\mathbf{v}$ is twice the size of $\mathbf{v}$, and $\mathbf{v}/2$ is half.

Scalar multiplication has the properties:

- $a(b\mathbf{v}) = (ab)\mathbf{v}$ [Scalar Association]
- $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$ [Scalar Distribution]
- $a(\mathbf{v} + \mathbf{w}) = a\mathbf{v} + a\mathbf{w}$ [Vector Distribution]

Scalar multiplication is often used to interpolate positions between two points P and Q. To get an intermediate point R part way from P to Q, given by a ratio $r \in [0,1]$, one scales the vector $\mathbf{v} = Q - P$ to $r\mathbf{v}$, and adds it to P to get: $R = P + r\mathbf{v} = P + r(Q - P) = (1-r)P + rQ$. Intermediate points are easily calculated; for example, to get the midpoint M between P and Q, use $r = ½$ to compute it as $(P+Q)/2$.



This interpolation equation is also used to represent the line through P and Q as a function of a parameter (see the section on “Geometric Applications” for “Lines”), namely as: $P(t) = (1 - t)P + tQ$ which is the ***line parametric equation***.

Affine Addition

We have already seen that the difference between two points can be considered as a vector. However, in general, it makes no sense to add two points together. Points denote an absolute position in space independent of any coordinate system describing them. Blindly adding individual coordinates together would give different answers for different coordinate reference frames.

Nevertheless, there is one special case, known as **affine addition**, where one can add points together as a weighted sum. In fact, in the previous section on scalar multiplication, we did just that to represent the points on a line going through two fixed points P and Q. More generally, given m points $P_0, \ldots, P_{m-1}$, one can define:

Definition
Affine Sum

$$P = \sum_{i=0}^{m-1} a_i P_i \quad \text{where} \quad \sum_{i=0}^{m-1} a_i = 1$$

where the coefficients must sum to 1. One interprets this sum as the center of mass of weights a_i located at the points P_i. And the center of mass is uniquely determined as the same point regardless of what coordinate frame of reference is being used.

For example, given equal weights $a_0 = a_1 = ½$, we have that $P = ½P_0 + ½P_1 = (P_0 + P_1)/2$ is always the midpoint of the line segment from P_0 to P_1. Further, every point on the line through P_0 and P_1 is uniquely represented by a pair (a_0,a_1) with $a_0 + a_1 = 1$. Similarly, the affine sum of three non-collinear points P_0, P_1, P_2, defines a point in the unique plane going through these points. And also, every point P in the plane $P_0P_1P_2$, is uniquely represented by a triple (a_0, a_1, a_2) with $a_0 + a_1 + a_2 = 1$. This triple is called the ***barycentric coordinate*** of its associated point P on the plane defined by the triangle of points $P_1P_2P_3$. Further, by substituting $a_1 = s$, $a_2 = t$, and $a_0 = 1 – s – t$, one can write the ***plane parametric equation*** to be: $P(s,t) = (1 – s – t)P_0 + sP_1 + tP_2 = P_0 + s(P_1 – P_0) + t(P_2 – P_0) = P_0 + s\mathbf{u} + t\mathbf{v}$, where $\mathbf{u} = (P_1 – P_0)$ and $\mathbf{v} = (P_2 – P_0)$ are independent vectors spanning the plane. The pair (s,t) is the ***parametric coordinate*** of P on the plane, and there is a unique parametric coordinate pair for each point on the plane.

Vector Length

The **length** of a vector is defined as:

Definition
Vector Length

$$|\mathbf{v}| = \sqrt{\sum_i v_i^2}$$

This gives the standard Euclidean geometry (Pythagorean) length for the line segment representing a vector. For a 2D vector $\mathbf{v} = (v_1, v_2)$, one has: $|\mathbf{v}|^2 = v_1^2 + v_2^2$, which is the Pythagorean theorem.

It is easy to prove the formula:

- $|a\mathbf{v}| = |a|\,|\mathbf{v}|$

showing that scalar multiplication really does scale the length of a vector as one would expect.

A **unit vector** is one whose length = 1. One can scale any vector $\mathbf{v}$ to get a unit vector $\mathbf{u}$ that points in the same direction as $\mathbf{v}$ by computing: $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$, and thus $|\mathbf{u}| = 1$. The process of scaling $\mathbf{v}$ to a unit vector $\mathbf{u}$ is called ***normalization***, and one says that $\mathbf{v}$ has been normalized. One thinks of $\mathbf{u}$ as the direction of $\mathbf{v}$ since $\mathbf{v} = |\mathbf{v}|\,\mathbf{u}$ simply scales $\mathbf{u}$ to the magnitude $|\mathbf{v}|$.

Vector Products

The Dot Product

The **dot product** (aka **inner product** or **scalar product**) of two vectors, $\mathbf{v}$ and $\mathbf{w}$, is defined as the (scalar) real number given by the sum of the products of their corresponding coordinates. This operation is denoted by a dot, "·", and given by the equation:

Definition Dot Product

$$\mathbf{v} \cdot \mathbf{w} = \sum_{i=1}^{n} v_i w_i$$

For example, if $\mathbf{v}$ and $\mathbf{w}$ are 2D vectors, then: $\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2$.

The dot product has the properties:

- $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$ [Vector Length]
- $(a\mathbf{v}) \cdot (b\mathbf{w}) = (ab)(\mathbf{v} \cdot \mathbf{w})$ [Scalar Association]
- $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$ [Commutation]
- $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) + (\mathbf{u} \cdot \mathbf{w})$ and
  $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = (\mathbf{u} \cdot \mathbf{w}) + (\mathbf{v} \cdot \mathbf{w})$ [Additive Distribution]

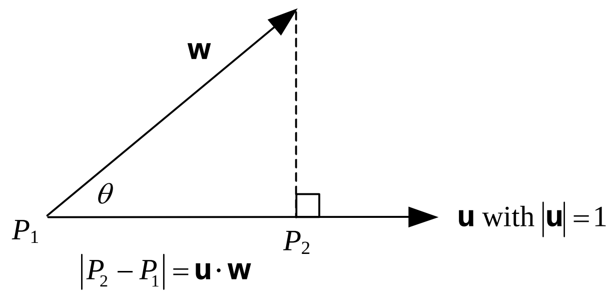
An amazing mathematical formula for the dot product is:

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}||\mathbf{w}|\cos\theta$$

where θ is the angle between the vectors $\mathbf{v}$ and $\mathbf{w}$. This formula is used extensively in computer graphics since it speeds up computation in many situations by avoiding direct usage of an inefficient trigonometric function. It is useful to note that when these are unit vectors with $|\mathbf{v}| = |\mathbf{w}| = 1$, then $\mathbf{v} \cdot \mathbf{w} = \cos\theta$. More generally, to compute $\cos\theta$ of the angle θ between two vectors, the following formula is convenient:

- $\cos\theta = \dfrac{\mathbf{v}}{|\mathbf{v}|} \cdot \dfrac{\mathbf{w}}{|\mathbf{w}|}$ = the dot product of the two normalized unit vectors.

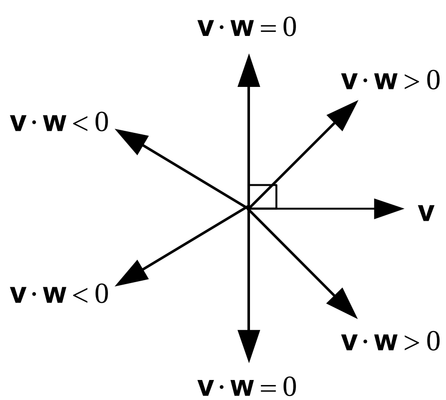
Additionally, the dot product formula can be interpreted geometrically as the projection of one vector onto the other. So, if $\mathbf{u}$ is a unit vector, then $\mathbf{u} \cdot \mathbf{w}$ is the length of the perpendicular projection of $\mathbf{w}$ onto $\mathbf{u}$, as shown in the diagram:



Further, when two vectors **v** and **w** are perpendicular, they are said to be ***normal*** to each other, and this is equivalent to their dot product being zero, that is: $\mathbf{v} \cdot \mathbf{w} = 0$. So this is a very simple and efficient test for perpendicularity. Because of this, for any vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ one can easily construct perpendicular vectors by zeroing all components except 2, flipping those two, and reversing the sign of one of them; for example, $(-v_2, v_1, 0, \ldots, 0)$, $(0, -v_3, v_2, 0, \ldots, 0)$, etc. The dot product of any of these with the original vector **v** is always = 0, and thus they are all perpendicular to **v**. For example, in 3D space with $\mathbf{v} = (v_1, v_2, v_3)$, the two vectors $\mathbf{u}_1 = (-v_2, v_1, 0)$ and $\mathbf{u}_2 = (0, -v_3, v_2)$, when nonzero, are a basis for the unique plane through the origin and perpendicular to **v**.

Beyond this, another important and useful consequence of the dot product cosine formula is that (for $|\theta| \leq 180^\circ$):

$$\mathbf{v} \cdot \mathbf{w} = 0 \iff \mathbf{v} \text{ and } \mathbf{w} \text{ are perpendicular; i.e. } |\theta| = 90^\circ$$
$$\mathbf{v} \cdot \mathbf{w} > 0 \iff \theta \text{ is an acute angle; i.e. } |\theta| < 90^\circ$$
$$\mathbf{v} \cdot \mathbf{w} < 0 \iff \theta \text{ is an obtuse angle; i.e. } |\theta| > 90^\circ$$



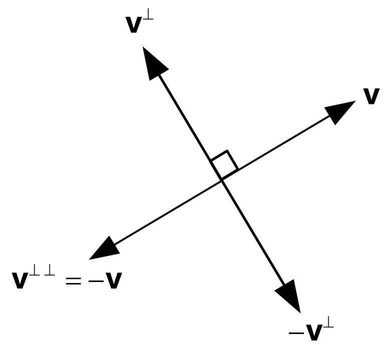
The 2D Perp Operator

Based on the preceding, we can define an operator on the 2D plane that gives a *counterclockwise* (ccw) normal (i.e.: perpendicular) vector of $\mathbf{v}$ to be:

Definition
2D Perp Operator

$$\mathbf{v}^{\perp} = (v_1, v_2)^{\perp} = (-v_2, v_1)$$

This operator, denoted by "$^{\perp}$", is called the **perp operator**. The **perp vector** $\mathbf{v}^{\perp}$ is the normal vector pointing to the left (ccw) side of the vector $\mathbf{v}$ as shown in the diagram:



Some properties of the **perp operator** are:

- $\mathbf{v}^{\perp} \cdot \mathbf{v} = 0$ [Perpendicular]
- $\left|\mathbf{v}^{\perp}\right| = \left|\mathbf{v}\right|$ [Preserves length]
- $(a\mathbf{v} + b\mathbf{w})^{\perp} = a\mathbf{v}^{\perp} + b\mathbf{w}^{\perp}$ [Linearity]
- $\mathbf{v}^{\perp\perp} = (\mathbf{v}^{\perp})^{\perp} = -\mathbf{v}$ [Antipotent]

The 2D Perp Product

Also in 2D space, there is another useful scalar product of two vectors $\mathbf{v}$ and $\mathbf{w}$, the **perp product** "$\perp$" (aka the **2D exterior product**, or **outer product)**, defined by:

Definition
Perp Product

$$\mathbf{v} \perp \mathbf{w} = \mathbf{v}^{\perp} \cdot \mathbf{w} = v_1 w_2 - v_2 w_1 = \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

where the 2x2 determinant is given by: $\begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc$.

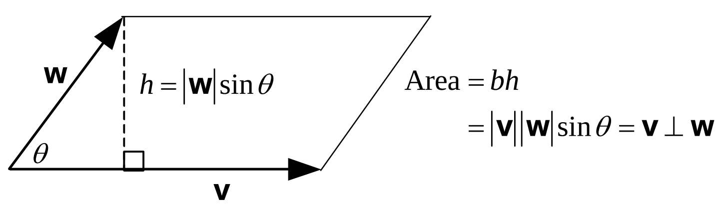
Some properties of the 2D perp product are:

- $\mathbf{v} \perp \mathbf{v} = 0$ [Nilpotent]
- $(a\mathbf{v}) \perp (b\mathbf{w}) = (ab)(\mathbf{v} \perp \mathbf{w})$ [Scalar Association]
- $\mathbf{v} \perp \mathbf{w} = -\mathbf{w} \perp \mathbf{v}$ [Antisymmetric]
- $\mathbf{u} \perp (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \perp \mathbf{v}) + (\mathbf{u} \perp \mathbf{w})$ and
  $(\mathbf{u} + \mathbf{v}) \perp \mathbf{w} = (\mathbf{u} \perp \mathbf{w}) + (\mathbf{v} \perp \mathbf{w})$ [Additive Distribution]
- $(\mathbf{v} \perp \mathbf{w})^2 + (\mathbf{v} \cdot \mathbf{w})^2 = |\mathbf{v}|^2 |\mathbf{w}|^2$

Also, for the 2D perp product, we have the formula:

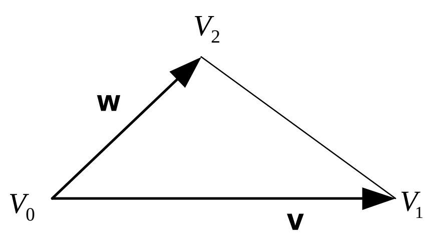
$$\mathbf{v} \perp \mathbf{w} = |\mathbf{v}||\mathbf{w}|\sin\theta$$

which can be used to compute $\sin\theta$ from $\mathbf{v}$ and $\mathbf{w}$. Further, geometrically the perp product gives the (signed) area of the 2D parallelogram spanned by $\mathbf{v}$ and $\mathbf{w}$, as shown in the diagram:



To compute the area of a 2D triangle with vertices V_0, V_1, V_2 and $V_i = (x_i, y_i)$ for $i = 0,2$, define the edge vectors $\mathbf{v} = V_1 - V_0$ and $\mathbf{w} = V_2 - V_0$. Then, since a triangle is half of a parallelogram, we can compute the (signed) area of $\Delta V_0V_1V_2$ as:

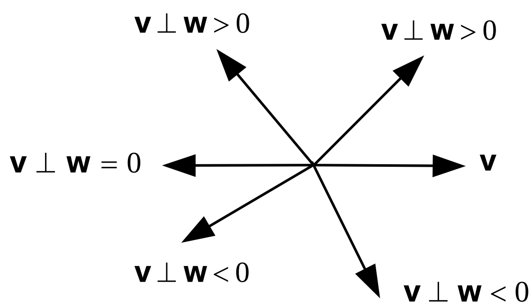
$$Area\ (\Delta_{2D}) = \tfrac{1}{2}(\mathbf{v} \perp \mathbf{w}) = \tfrac{1}{2}\begin{vmatrix} (x_1 - x_0) & (x_2 - x_0) \\ (y_1 - y_0) & (y_2 - y_0) \end{vmatrix}$$



which is a very efficient computation. This signed area is positive when the vertices V_0, V_1, V_2 are oriented counterclockwise and is negative when they are oriented clockwise, so it can also be used to test for the orientation of a triangle. This is the same as testing for which side of the directed line through V_0V_1 the point V_2 lies on: it is left of V_0V_1 when the area is positive, on the line when the area = 0, and on the right side when the area is negative.

The 2D perp product can also be used to determine which side (left or right) of one vector another vector is pointing, since (for $|\theta| \le 180°$):

$$\mathbf{v} \perp \mathbf{w} = 0 \iff \mathbf{v} \text{ and } \mathbf{w} \text{ are colinear; i.e. } |\theta| = 0 \text{ or } 180°$$
$$\mathbf{v} \perp \mathbf{w} > 0 \iff \mathbf{w} \text{ is left of } \mathbf{v}\text{; i.e. } 0 < \theta < 180°$$
$$\mathbf{v} \perp \mathbf{w} < 0 \iff \mathbf{w} \text{ is right of } \mathbf{v}\text{; i.e. } 0 > \theta > -180°$$



The 3D Cross Product

The **3D cross product** (aka **3D outer product** or **vector product**) of two vectors, $\mathbf{v}$ and $\mathbf{w}$, is only defined for 3D vectors, say $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$. It is denoted by a cross, "×", and is defined by the equation:

Definition
Cross Product

$$\mathbf{v} \times \mathbf{w} = \left(\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}, \begin{vmatrix} v_3 & v_1 \\ w_3 & w_1 \end{vmatrix}, \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \right) \quad \text{where} \quad \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc$$

The cross product has the properties:

- $\mathbf{v} \times \mathbf{v} = \mathbf{0} = (0,0,0)$ [Nilpotent]
- $(a\mathbf{v}) \times (b\mathbf{w}) = (ab)(\mathbf{v} \times \mathbf{w})$ [Scalar Association]
- $\mathbf{w} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{w})$ [Antisymmetric]
- $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$ and
  $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$ [Additive Distribution]
- $(\mathbf{v} \times \mathbf{w}) \cdot \mathbf{v} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{w} = 0$ [Perpendicularity]
- $|\mathbf{v} \times \mathbf{w}|^2 + (\mathbf{v} \cdot \mathbf{w})^2 = |\mathbf{v}|^2 |\mathbf{w}|^2$

However, the cross product is not associative with itself, and it is not distributive with the dot product. Instead, one has the following formulas. These are not often used in computer graphics, but sometimes can streamline computations. Note that the formulas for left and right association are different.

- $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\, \mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\, \mathbf{u}$ [Left Association]
- $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\, \mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\, \mathbf{w}$ [Right Association]
- $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ [Dot-Cross Association]
- $(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{w} \times \mathbf{x}) = (\mathbf{u} \cdot \mathbf{w})(\mathbf{v} \cdot \mathbf{x}) - (\mathbf{v} \cdot \mathbf{w})(\mathbf{u} \cdot \mathbf{x})$ [Lagrange's Identity]

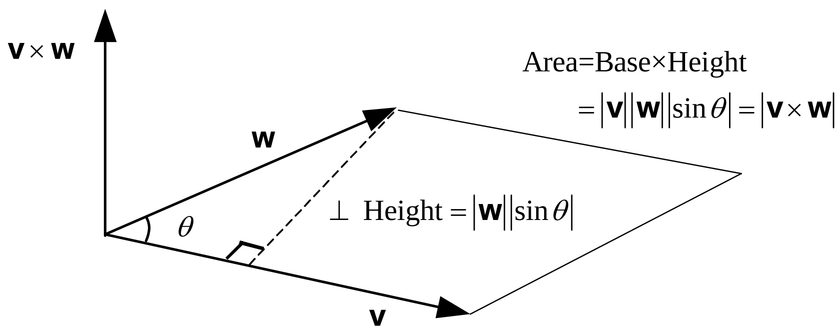
From Lagrange's Identity, we can also compute that:

$$\begin{aligned} |\mathbf{u} \times \mathbf{v}|^2 &= (\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{u} \times \mathbf{v}) \\ &= (\mathbf{u} \cdot \mathbf{u})(\mathbf{v} \cdot \mathbf{v}) - (\mathbf{u} \cdot \mathbf{v})^2 \\ &= |\mathbf{u}|^2 |\mathbf{v}|^2 \, (1 - \cos^2 \theta) \\ &= |\mathbf{u}|^2 |\mathbf{v}|^2 \, \sin^2 \theta \end{aligned}$$

which demonstrates the important cross product formula:

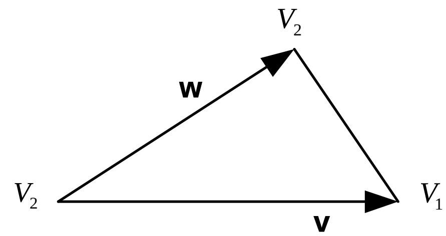
$$\begin{aligned} |\mathbf{v} \times \mathbf{w}| &= |\mathbf{v}||\mathbf{w}||\sin \theta| \\ \mathbf{v} \times \mathbf{w} &= \left(|\mathbf{v}||\mathbf{w}| \sin \theta\right) \mathbf{u} \quad \text{with } |\mathbf{u}| = 1 \end{aligned}$$

where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$. Note that $\mathbf{u}$ is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$. Geometrically, the cross product points outward from the **vw**-plane using a right-hand rule; and its magnitude is the area of the parallelogram spanned by $\mathbf{v}$ and $\mathbf{w}$ as shown in the following diagram:



This fact makes the cross product very useful for doing 3D area computations. For example, to get the area of a 3D triangle $\Delta_{3D} = \Delta V_0 V_1 V_2$ with vertices V_0, V_1, V_2, define the edge vectors $\mathbf{v} = V_1 - V_0$ and $\mathbf{w} = V_2 - V_0$, and compute the area of $\Delta V_0 V_1 V_2$ to be:

$$\text{Area}\ (\Delta_{3D}) = \frac{1}{2}\left|\mathbf{v}\times\mathbf{w}\right| = \frac{1}{2}\left|\left(V_1 - V_0\right)\times\left(V_2 - V_0\right)\right|$$



Another important consequence of the cross product formula is that if **v** and **w** are perpendicular unit vectors, then **v** × **w** is also a unit vector since $\sin\theta = 1$. Thus, the three vectors **v**, **w**, and **v** × **w** form an orthogonal coordinate frame of reference (or basis) for 3D space. This is used in 3D graphics to simplify perspective calculations from observer viewpoints.

Finally, in 2D space, there is a relationship between the embedded cross product and the 2D perp product. One can embed a 2D vector **v** = (v_1, v_2) in 3D space by appending a third coordinate equal to 0, namely: (**v**,0) = $(v_1, v_2, 0)$. Then, for two vectors **v** and **w**, the embedded 3D cross product is: (**v**,0) × (**w**,0) = $(0, 0, v_1w_2 - v_2w_1)$ = (0, 0, **v** ⊥ **w**), which only has a single non-zero third component that is equal to the perp product **v** ⊥ **w**.

The 3D Triple Product

Another useful geometric computation is the **triple product:**

Definition
Triple Product

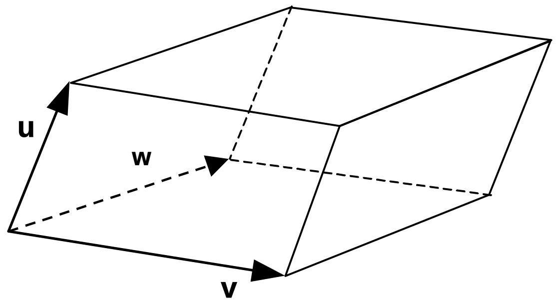
$$\left[\mathbf{u}\,\mathbf{v}\,\mathbf{w}\right] = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$$

which satisfies the equation:

$$\left[\mathbf{u}\,\mathbf{v}\,\mathbf{w}\right] = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

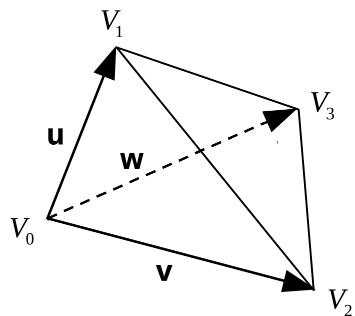
The triple product is equal to the volume of the parallelepiped (the 3D analogue of a parallelogram) defined by the three vectors **u, v** and **w** starting from the same corner point as shown in the diagram:

$$\text{Volume}(\square_{3D}) = [\mathbf{u}\ \mathbf{v}\ \mathbf{w}] = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$$



From this we can get the volume of a 3D tetrahedron with vertices V_0, V_1, V_2, V_3 given by $V_i = (x_i, y_i, z_i)$ for i=0,3. The volume of this tetrahedron is 1/6 that of the parallelepiped spanned by the vectors $\mathbf{u} = V_1 - V_0$, $\mathbf{v} = V_2 - V_0$, and $\mathbf{w} = V_3 - V_0$. This gives:

$$\begin{aligned}\text{Volume }(\Delta V_0V_1V_2V_3) &= \frac{1}{6}(V_1 - V_0)\cdot\left[(V_2 - V_0)\times(V_3 - V_0)\right] \\ &= \frac{1}{6}\begin{vmatrix}(x_1 - x_0) & (y_1 - y_0) & (z_1 - z_0) \\ (x_2 - x_0) & (y_2 - y_0) & (z_2 - z_0) \\ (x_3 - x_0) & (y_3 - y_0) & (z_3 - z_0)\end{vmatrix}\end{aligned}$$



Geometric Applications

Lines

In any dimension, a line is uniquely defined by any two points P_0 and P_1 on it. The line consists of all points satisfying the linear ***parametric equation***:

$$P(t) = (1-t)\,P_0 + t\,P_1, \text{ where } t \text{ is a real number.}$$

Each value of t corresponds to one point $P(t)$ on the line, for example $P(0) = P_0$ and $P(1) = P_1$, where P_0 and P_1 are the two points that defined the line. So, the (finite) line segment bounded by the endpoints P_0 and P_1 is given by the points $P(t)$ with $0 \le t \le 1$. Also, the line equation can be rewritten as:

$$P(t) = P_0 + t\,(P_1 - P_0) = P_0 + t\,\mathbf{v}, \text{ where } \mathbf{v} = (P_1 - P_0)$$

representing the line as starting at the point P_0 and extending in the direction of the vector $\mathbf{v}$.

Distance of a Point to a Line

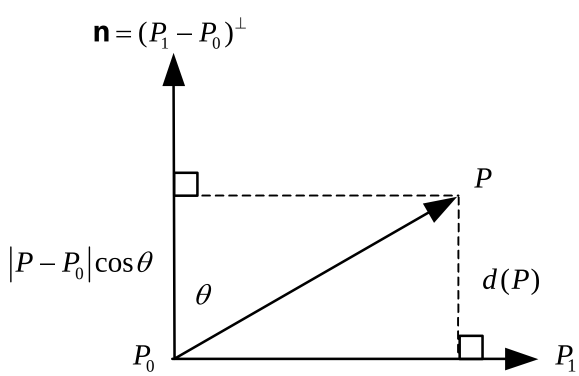
From the properties of the dot, perp, and cross products, one can do many useful computations. One of these is computing the distance from a point to a line in 2D and 3D. There are a number of alternative ways to compute this distance, depending on how the line is represented and the dimension of the space. Here, we let the line $\mathbf{L}$ be given by two points, P_0 and P_1, through which it passes, and $d(P, \mathbf{L})$ be the perpendicular distance from a point P to the line $\mathbf{L}$.

2D Case:

In the 2D plane we can use the dot product to compute $d(P, \mathbf{L})$. A normal vector to the line is given by $\mathbf{n} = (P_1 - P_0)^{\perp}$. Then, $d(P, \mathbf{L})$ is equal to the length of the projection of the vector from P_0 to P onto the normal $\mathbf{n}$ as shown in the diagram. Computing this we get:

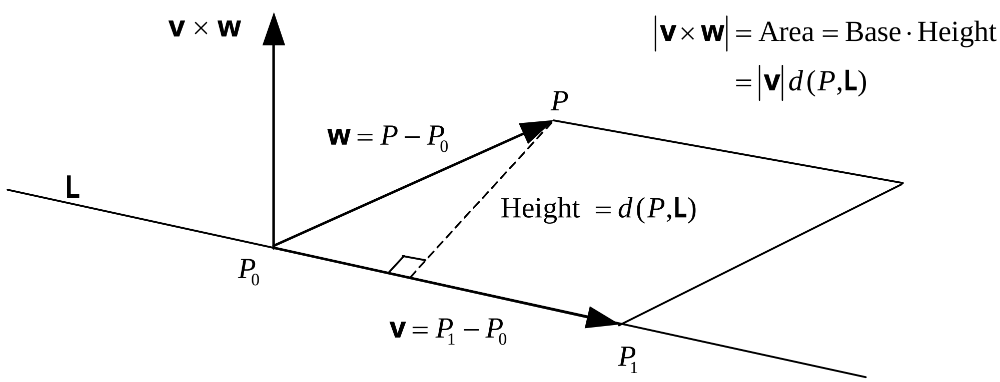
$$d(P,\mathbf{L}) = \left|P - P_0\right|\cos\theta = \frac{\mathbf{n}}{|\mathbf{n}|}\cdot(P - P_0) = \frac{\Delta y(x - x_0) - \Delta x(y - y_0)}{\sqrt{\Delta x^2 + \Delta y^2}}$$

where $P = (x, y)$, $P_0 = (x_0, y_0)$, $P_1 = (x_1, y_1)$, $\Delta x = (x_1 - x_0)$, and $\Delta y = (y_1 - y_0)$. Note that we have computed this as a *signed* value that is positive when P is on the left side of the directed line P_0P_1, and negative when on the right side.



3D Case:
In 3D space the cross product can be used to compute $d(P, \mathbf{L})$. Consider the parallelogram defined by the vectors $\mathbf{v} = P_1 - P_0$, and $\mathbf{w} = P - P_0$. Its area is equal to $|\mathbf{v} \times \mathbf{w}|$. But also, viewing $\mathbf{v}$ as the base of the parallelogram and $d(P, \mathbf{L})$ as its height, then its area is equal to the base times the height, namely $|\mathbf{v}|\, d(P,\mathbf{L})$. Equating these and solving, one gets:

$$d(P,\mathbf{L}) = \frac{|\mathbf{v} \times \mathbf{w}|}{|\mathbf{v}|} = \frac{|(P_1 - P_0) \times (P - P_0)|}{|P_1 - P_0|}$$



However, unlike the 2D case, this is an absolute *unsigned* value, and there is no sense in saying that a point is to the left or right of a line in 3D space. The equivalent concept in 3D space would be the distance of a point to a plane, since a plane divides 3D space into two disjoint pieces, and one can compute the distance from a point to a plane as a signed quantity.

Summary

Notation

Scalars	*lower case*	*italic*	$a, b, ..., x, y, ...$
Points	*UPPER CASE*	*ITALIC*	$P = (p_1, p_2, \ldots, p_n)$
Vectors	**lower case**	**bold gothic**	$\mathbf{v} = (v_1, v_2, \ldots, v_n)$

Definitions

Point Difference: $\mathbf{v} = Q - P = (q_i - p_i)$

Vector Addition: $\mathbf{v} \pm \mathbf{w} = (v_i \pm w_i)$

Scalar Multiplication: $a\mathbf{v} = (av_i)$

Dot Product: $\mathbf{v} \cdot \mathbf{w} = \sum_i v_i w_i$

Vector Length: $|\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{v} = \sum_i v_i^{\,2}$

2D Perp Operator: $\mathbf{v}^{\perp} = (v_1, v_2)^{\perp} = (-v_2, v_1)$

2D Perp Product: $\mathbf{v} \perp \mathbf{w} = \mathbf{v}^{\perp} \cdot \mathbf{w} = v_1 w_2 - v_2 w_1 = \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$

3D Cross Product: $\mathbf{v} \times \mathbf{w} = \left(\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}, \begin{vmatrix} v_3 & v_1 \\ w_3 & w_1 \end{vmatrix}, \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \right)$ where $\begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc$

3D Triple Product: $[\mathbf{u} \;\; \mathbf{v} \;\; \mathbf{w}] = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 \end{vmatrix}$

Basic Equations

Dot Product: $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}||\mathbf{w}| \cos\theta$

$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) + (\mathbf{u} \cdot \mathbf{w})$

2D Perp Product: $\mathbf{v} \perp \mathbf{w} = |\mathbf{v}||\mathbf{w}| \sin\theta$

$\mathbf{u} \perp (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \perp \mathbf{v}) + (\mathbf{u} \perp \mathbf{w})$

$(\mathbf{v} \perp \mathbf{w})^2 + (\mathbf{v} \cdot \mathbf{w})^2 = |\mathbf{v}|^2 |\mathbf{w}|^2$

3D Cross Product: $|\mathbf{v} \times \mathbf{w}| = |\mathbf{v}||\mathbf{w}||\sin\theta|$

$\mathbf{w} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{w})$

$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$

$(\mathbf{v} \times \mathbf{w}) \cdot \mathbf{v} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{w} = 0$

Exercises

(1) Let **v** = (8,6). Compute the length of **v**. Give the coordinates of a ***unit*** vector **u** ***normal*** to **v**.

(2) For two points $P = (1,2)$ and $Q = (4,8)$, compute the intermediate points that are 1/3, 1/2 and 3/4 the way between P and Q.

(3) Let **v** = (8,6), **w** = (0,5), and θ = the angle between **v** and **w**. Using the dot and perp products, compute $\cos(\theta)$ and $\sin(\theta)$.

(4) Let $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ be any two 3D vectors. Prove that the equations: $(\mathbf{v} \times \mathbf{w}) \cdot \mathbf{v} = 0$ and $(\mathbf{v} \times \mathbf{w}) \cdot \mathbf{w} = 0$ are always true, and thus that $\mathbf{v} \times \mathbf{w}$ is always perpendicular to both **v** and **w**.

(5) Let $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ be any three 3D vectors. Prove that the equation: $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ is always true. Show, using geometric reasoning, that it is equal to the volume of the 3D parallelepiped defined by edge vectors **u**, **v,** and **w** starting at the origin.

(6) Let a 2D (infinite) line pass through two points P_0 and P_1. Given any arbitrary point P in the plane, show that the perp product: $(P - P_0) \perp (P_1 - P_0)$ will be positive for points P on one side of the line, and negative for points on the other side of the line.

(7) Using the previous exercise, develop a test for whether a ***finite segment*** between points Q_0 and Q_1 crosses (i.e. intersects) the (infinite) line through P_0 and P_1. Use this to develop another test for whether the ***finite segment*** Q_0Q_1 intersects with the ***finite segment*** P_0P_1 without actually computing the point of intersection.

(8) Let a 3D (infinite) line **L** be defined by two points $P_0 = (x_0, y_0, z_0)$ and $P_1 = (x_1, y_1, z_1)$ on it, and let $P = (x, y, z)$ be an arbitrary point. Put $\Delta x = (x_1 - x_0)$, $\Delta y = (y_1 - y_0)$, $\Delta z = (z_1 - z_0)$; and $\Delta r = (x - x_0)$, $\Delta s = (y - y_0)$, $\Delta t = (z - z_0)$. Show that the distance $d(P, \mathbf{L})$ from P to the line **L** is given by the formula:

$$d(P,\mathbf{L})^2 = \frac{(\Delta z\Delta s - \Delta y\Delta t)^2 + (\Delta x\Delta t - \Delta z\Delta r)^2 + (\Delta y\Delta r - \Delta x\Delta s)^2}{(\Delta x^2 + \Delta y^2 + \Delta z^2)}$$

(9) In the 2D plane, let **a** and **b** be two fixed perpendicular vectors, and let **v** be any arbitrary vector. Show that **v** can be decomposed into the sum of two vectors in the directions of **a** and **b**. That is, show that the linear equation $\mathbf{v} = \alpha\mathbf{a} + \beta\mathbf{b}$ can be solved by finding explicit formulas for the coefficients α and β in terms of **a, b** , and **v**. [Hint: use the dot product.]

(10) Using the preceding exercise, in the 2D plane, show how to compute the reflection of a vector **v** off a fixed linear mirror. Write an explicit formula for the reflected vector. [Hint: let **a** be a vector along the mirror's surface, and let **b** be perpendicular to the mirror.] Note that the vector **v** could represent the direction of travel of anything from a light ray to a billiard ball that can reflect off a straight line obstacle.

(11) In 3D space, three distinct noncollinear points P_0, P_1, and P_2 define a plane π. Let P be any 3D point, and $d(P,\pi)$ be the perpendicular distance from P to the plane π. Derive a formula to compute $d(P,\pi)$. [Hint: use a vector **n** normal to the plane π].

(12) In 3D space, let π be a plane through the points P_0, P_1, and P_2; and let **L** be a line through the points Q_0 and Q_1. Using vector operations, derive a formula to compute the intersection of **L** with π.