Physical parameter sensitivity of system eigenvalues and physical model reduction

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Abstract

The identification of subsystems and/or components that is related to a given eigenvalue of the overall system is a challenging and important topic. The use of special structure of the system matrices obtained busing bond graphs can result in identifying subsystems and/or components that affect a given eigenvalue of an overall system. This paper, by making use of a set of theorems and definitions proposes an efficient procedure for this purpose. The basic procedure is based upon the calculation of sensitivity of eigenvalues. The so-called "effect" matrices are produced that indicates the relative importance of physical parameters on a selected eigenvalue. In addition to the relative importance, the effect matrix is used for an efficient physical model reduction procedure. Furthermore, reasons of different dynamic behavior of a system can be explained. Use of effect matrices also improves the physical model reduction method based on decomposition procedures. Three examples are given to illustrate the approach and its consequences.
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1. Introduction

The topic of model reduction has been studied for many years now, and many methods have been suggested for obtaining suitable low-order approximations [1–4]. Most of the techniques in the literature take into account a criterion for the 'goodness' of the reduced model [1,4]. For example, the balancing approach of Moore [1] uses coordinate transformations to convert the system to a special balanced form from which a reduced model can be obtained. In this method, Hankel singular values (HSVs) are used as the criterion for the error of the reduced-order model.

In addition to the purely 'numerical' procedures above, a useful type of reduced model is obtained by removing some physical components from the original model. This approach is known as model reduction in physical domain. In these methods either the components associated with small power flow are eliminated as they have small contribution to the dynamic behavior of a system [5] or decomposition of physical systems that are useful for the identification of dominant components or subsystems are utilized. Orbak et al. [6]. Orbak et al. [6] uses the bond graph formulation for a decomposition based physical model reduction method. As discussed in [7], decomposition procedures may fail to identify relevant components when the system has uniform parameters or when the loop gains cannot be distinguished.

On the other hand, sensitivity of a dynamic system to its physical parameters is often of interest. In this context, eigenvalue and eigenvector derivatives have been useful for determining the sensitivity of dynamic response to system parameter variations. For example, knowledge of the eigenvector derivatives with respect to physical parameters can be used to optimize a structural design or minimize its sensitivity to parameters. Such information can be used regularly for structural optimization, and for the improvement of the agreement between analytical and experimental results [8,9]. Furthermore, eigen derivatives can be directly applied to system identification and robust performance tests for structural control systems [8,9]. On the other hand, the eigenvalue sensitivity with respect to a physical parameter gives an estimate of the eigenvalue shift when such parameter is changed. Thus, eigenvalue sensitivity may also be used to obtain reduced-order models of physical systems.

Several methods have been proposed to analyze the connection between a system variable and its modes [8–10,15,16]. One of these methods, the participation factor approach, has been extensively used for the analysis of power systems [10,11]. Dynamic systems with large number of state variables, such as power systems, are often too complex to be analyzed. The physical knowledge of the system might be utilized to simplify the model: knowing that the system presents an oscillatory behavior, the interest might be focused on a particular system eigenvalue (mode), by looking for the physical state variables most involved in the oscillation. In such cases, the participation factors might be useful in exploring the state variables that are relevant in the evolution of a particular eigenvalue [12].

Since the participation factors can be used to detect the states that are most involved in an eigenvalue, it is clear that once the eigenvalues of interest are identified, participation factors might help to obtain a reduced-order model which still conserves the most relevant dynamics of the system. In particular, the most important problem is the identification of subsystems/components that affect a given eigenvalue.

In some cases, the reconstruction of a matrix from prescribed spectral data that may consist of the complete or only partial information of eigenvalues and/or eigenvectors becomes important. In such cases, inverse eigenvalue problems are formed and solved [15,16]. The objective of inverse eigenvalue problems is to reconstruct the physical parameters of a certain system from the knowledge or desire of its dynamical behavior. Since the dynamical behavior often is governed by the underlying natural frequencies and/or normal modes, the spectral constraints are thus imposed. For this design approach the sensitivities of the inverse of the system matrix are used [16].

In this paper, a solution to the problem of sensitivity of a physical system with respect to its physical parameters is provided. For this purpose, in addition to a general analysis of participation factors and its relation to residues and eigenvalues, the use of eigenvalue sensitivity using special state-space descriptions has been investigated. The use of system matrices with certain structures lead to clearer and simpler results. Analyzing the procedures for obtaining system matrices using bond graphs leads to a very efficient solution as discussed by Rosenberg [13]. Thus in the following sections, the state-space representation discussed above will be briefly summarized. Then, the calculation of eigenvalue sensitivities using eigenvectors and their relationship to participation factors will be analyzed. Based on this analysis "effect" matrices are introduced that indicates the relative importance of physical parameters on selected eigenvalues. The effect matrices given in this section not only identify the irrelevant components but also give a relative measure of their contribution to the eigenvalue. The use of effect matrices also overcomes the problem of uniform parameters of the method in Orbak et al. [6].

2. Structured representation of LTI systems using bond graphs

A linear physical system can be characterized with matrices that identify the components and that define the structure of a system [13,14]. The parameters of the components can be described by two matrices, one for independent energy storage elements and one for dissipation elements. The energy storage elements can be represented by the matrix $\mathbf{S}$, defined as

$$\mathbf{z} = \mathbf{S}\mathbf{x} \tag{1}$$

where $\mathbf{x}$ is the generalized momentum/displacement vector and $\mathbf{z}$ is the corresponding flow/effort vector. In these vectors, x_i is the generalized momentum/displacement associated with the ith independent energy storage element (state) and z_i is the flow/

effort as the causal output of that element. For an LTI system with all of the independent energy storage elements of a total number of n are one port, $\mathbf{S}$ is a diagonal matrix of the form $\mathrm{diag}[s_1, s_2, \ldots, s_n]$, with s_i's as the parameters of the energy storage elements. From a computation point of view, if the ith independent energy storage element is a capacitance or an inertance, then $s_i = 1/C_i$ or $s_i = 1/I_i$, respectively.

On the other hand, the dissipation elements can be represented by the matrix $\mathbf{L}$, which contains the parameter values as follows: $\mathbf{d}_{\mathrm{out}} = \mathbf{L}\mathbf{d}_{\mathrm{in}}$, where d_{in_j} and d_{out_j} stand for the causal input and causal output of the jth dissipation element, respectively. Again, for an LTI system with m one port dissipation elements, $\mathbf{L}$ is diagonal and of the form $\mathrm{diag}[l_1, l_2, \ldots, l_m]$. When the jth dissipation component has a flow or an effort as the causal input and an effort or a flow as the causal output, then $l_j = R_j$ or $l_j = 1/R_j$, respectively.

Then, the structure of a system is described by [13]

$$\dot{\mathbf{x}} = \mathbf{J}_{SS}\mathbf{z} + \mathbf{J}_{SL}\mathbf{d}_{\mathrm{out}} + \mathbf{J}_{SU}\mathbf{u}, \tag{2}$$

$$\mathbf{d}_{\mathrm{in}} = \mathbf{J}_{LS}\mathbf{z} + \mathbf{J}_{LL}\mathbf{d}_{\mathrm{out}} + \mathbf{J}_{LU}\mathbf{u} \tag{3}$$

where, $\mathbf{J}_{ij}$ represents the connectivity matrix between the outputs of j elements to inputs of i elements. In this representation L, S and U refers to dissipation, energy storage and input variables, respectively.

As a result, the system's state-space equation can be rearranged as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{4}$$

where

$$\mathbf{A} = [\mathbf{J}_{SS} + \mathbf{J}_{SL}\mathbf{L}(\mathbf{I} - \mathbf{J}_{LL}\mathbf{L})^{-1}\mathbf{J}_{LS}]\mathbf{S} \equiv \mathbf{J}\mathbf{S}, \tag{5}$$

$$\mathbf{B} = \mathbf{J}_{SU} + \mathbf{J}_{SL}\mathbf{L}(\mathbf{I} - \mathbf{J}_{LL}\mathbf{L})^{-1}\mathbf{J}_{LU}. \tag{6}$$

3. Calculation of eigenvalue sensitivities

Consider the LTI continuous time system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{7}$$

where $\mathbf{x} \in \mathscr{R}^n$ and $\mathbf{A} \in \mathscr{R}^{n \times n}$.

Using modal decomposition matrix $\mathbf{A}$ can be written as

$$\mathbf{A} = \mathbf{U}\mathbf{\Lambda}\mathbf{V}$$

$$= \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \mathbf{v}_1^{\mathrm{T}} \\ \mathbf{v}_2^{\mathrm{T}} \\ \vdots \\ \mathbf{v}_n^{\mathrm{T}} \end{bmatrix} \tag{8}$$

where **U** and **V** are the right and left eigenvector matrices, and Λ is a diagonal eigenvalue matrix. It should be noted that the eigenvectors $\mathbf{u}_i$ and $\mathbf{v}_i$ can always be chosen so that $\mathbf{u}_i^T \mathbf{v}_i = 1$, or similarly in matrix form, $\mathbf{UV} = \mathbf{VU} = \mathbf{I}$. Furthermore, in this representation, the **A** matrix is assumed to have only distinct eigenvalues. This assumption is used throughout this paper for brevity. In case of repeated eigenvalues, the diagonal matrix becomes a Jordan form matrix, and the corresponding right and left eigenvectors will become generalized eigenvectors.

Using the modal decomposition with a given initial condition vector $\mathbf{x}(0)$, the solution of Eq. (7) is given as:

$$\mathbf{x}(t) = \sum_{i=1}^{n} e^{\lambda_i t} \mathbf{u}_i \mathbf{v}_i^T \mathbf{x}(0). \tag{9}$$

From this equation, one can write the kth state as follows:

$$
\begin{aligned}
\mathbf{x}^k(t) &= \sum_{i=1}^{n} e^{\lambda_i t} \mathbf{u}_i^k \mathbf{v}_i^T \mathbf{x}(0) \\
&= \sum_{i=1}^{n} e^{\lambda_i t} \mathbf{u}_i^k \left[\mathbf{v}_i^k \mathbf{x}^k(0) + \sum_{j=1, j \neq i}^{n} \mathbf{v}_i^j \mathbf{x}^j(0) \right] \\
&= \sum_{i=1}^{n} e^{\lambda_i t} p_{ki} \mathbf{x}^k(0) + \sum_{i=1}^{n} e^{\lambda_i t} \left[\sum_{j=1, j \neq k}^{n} P_{kij} \mathbf{x}^j(0) \right]
\end{aligned}
\tag{10}
$$

where

$$p_{ki} \triangleq \mathbf{u}_i^k \mathbf{v}_i^k \quad \text{participation factor,}$$
$$P_{kij} \triangleq \mathbf{u}_i^k \mathbf{v}_i^j \quad \text{generalized participation factor.} \tag{11}$$

The concept of participation factor was developed in [11] to measure the degree of participation of a state variable in a mode. Therefore, participation factor p_{ki} can be interpreted as the weight of the participation of ith mode in the kth state component. Simply, the participation factors can be seen as right eigenvectors weighted by left eigenvectors [11].

Using the participation values, a participation matrix can be formed as [11].

$$
\mathbf{H} = \begin{bmatrix}
p_{11} & p_{12} & \cdots & p_{1n} \\
p_{21} & p_{22} & \cdots & p_{2n} \\
\vdots & \vdots & \vdots & \vdots \\
p_{n1} & p_{n2} & \cdots & p_{nn}
\end{bmatrix}. \tag{12}
$$

For matrix **H**, and for generalized participation values, the following properties can be identified [12]:

(i) $\sum_{i=1}^{n} p_{ki} = 1$.
(ii) $\sum_{k=1}^{n} p_{ki} = 1$.
(iii) $\sum_{i=1}^{n} P_{kij} = 0$.

In addition to the above basic properties, the following theorems can be stated and proved.

Theorem 1. *The generalized participation values are considered as the sensitivities of the eigenvalues of the matrix* **A**

$$P_{kij} = \frac{\partial \lambda_i}{\partial a_{jk}} \tag{13}$$

where a_{jk} *represents the jkth element of matrix* **A** [12]*.*

Proof.

$$\mathbf{v}_i^T \mathbf{A} \mathbf{u}_i = \lambda_i \mathbf{v}_i^T \mathbf{u}_i = \lambda_i.$$

Then,

$$
\begin{aligned}
\frac{\partial \lambda_i}{\partial q} &= \frac{\partial (\mathbf{v}_i^T \mathbf{A} \mathbf{u}_i)}{\partial q} \\
&= \frac{\partial \mathbf{v}_i^T}{\partial q} \underbrace{\mathbf{A} \mathbf{u}_i}_{\lambda_i \mathbf{u}_i} + \mathbf{v}_i^T \frac{\partial (\mathbf{A} \mathbf{u}_i)}{\partial q} \\
&= \lambda_i \frac{\partial \mathbf{v}_i^T}{\partial q} \mathbf{u}_i + \mathbf{v}_i^T \left(\frac{\partial \mathbf{A}}{\partial q} \mathbf{u}_i + \mathbf{A} \frac{\partial \mathbf{u}_i}{\partial q} \right) \\
&= \lambda_i \frac{\partial \mathbf{v}_i^T}{\partial q} \mathbf{u}_i + \mathbf{v}_i^T \frac{\partial \mathbf{A}}{\partial q} \mathbf{u}_i + \underbrace{\mathbf{v}_i^T \mathbf{A}}_{\lambda_i \mathbf{v}_i^T} \frac{\partial \mathbf{u}_i}{\partial q} \\
&= \mathbf{v}_i^T \frac{\partial \mathbf{A}}{\partial q} \mathbf{u}_i + \lambda_i \frac{\partial \mathbf{v}_i^T}{\partial q} \mathbf{u}_i + \lambda_i \mathbf{v}_i^T \frac{\partial \mathbf{u}_i}{\partial q} \\
&= \mathbf{v}_i^T \frac{\partial \mathbf{A}}{\partial q} \mathbf{u}_i + \lambda_i \underbrace{\frac{\partial (\mathbf{v}_i^T \mathbf{u}_i)}{\partial q}}_{=0 \text{ as } \mathbf{v}_i^T \mathbf{u}_i = 1} \\
&= \mathbf{v}_i^T \frac{\partial \mathbf{A}}{\partial q} \mathbf{u}_i.
\end{aligned}
$$

If the parameter q is the element a_{jk} of the matrix **A**, then $\partial \mathbf{A} / \partial a_{jk}$ is a matrix whose elements are all zero and the element in jth row and kth column is one. Thus, one can write,

$$\frac{\partial \mathbf{A}}{\partial a_{jk}} = \mathbf{e}_j \mathbf{e}_k^T$$

where $\mathbf{e}_j$ and $\mathbf{e}_k$ are the jth and kth column of an identity matrix $\mathbf{I}_{n \times n}$, respectively. As a result,

$$\frac{\partial \lambda_i}{\partial a_{jk}} = \mathbf{v}_i^{\mathrm{T}} \mathbf{e}_j \mathbf{e}_k^{\mathrm{T}} \mathbf{u}_i = v_i^j u_i^k \equiv P_{kij}. \qquad \square$$

It should be noted that this proof directly leads to the result that participation factors are the sensitivities of the diagonal terms of $\mathbf{A}$, i.e.

$$p_{ki} = \frac{\partial \lambda_i}{\partial a_{kk}}.$$

From the descriptions above the following theorem can also be deduced.

Theorem 2. *The entries of the system matrix* $\mathbf{A}$ *can be expressed as a linear combination of the eigenvalues with the coefficients being the participation values* [12].

Proof. The $\mathbf{A}$ matrix can be written as

$$\mathbf{A} = \sum_{i=1}^{n} \lambda_i \mathbf{u}_i \mathbf{v}_i^{\mathrm{T}}$$

utilizing the dyadic form. Then,

$$a_{kj} = \mathbf{e}_k^{\mathrm{T}} \mathbf{A} \mathbf{e}_j$$

$$= \sum_{i=1}^{n} \lambda_i \underbrace{\mathbf{e}_k^{\mathrm{T}} \mathbf{u}_i}_{\mathbf{u}_i^k} \overbrace{\mathbf{v}_i^{\mathrm{T}} \mathbf{e}_j}^{v_i^j}$$

$$= \sum_{i=1}^{n} \lambda_i P_{kij}.$$

Specifically, for the diagonal elements,

$$a_{kk} = \sum_{i=1}^{n} \lambda_i p_{ki}$$

is obtained. $\square$

Furthermore, the following lemma can be written.

Lemma 3. *The following relation between the participation values* (P_{kij}) *and partial fraction expansion residues* (R_i) *holds:*

$$P_{kij} \triangleq \mathbf{e}_k^{\mathrm{T}} R_i \mathbf{e}_j.$$

Proof. As one can write,

$$(s\mathbf{I} - \mathbf{A})^{-1} = \sum_{i=1}^{n} \frac{R_i}{s - \lambda_i}$$

and

$$e^{\mathbf{A}t} = e^{\mathbf{U}\mathbf{\Lambda}\mathbf{V}t} = \mathbf{U}e^{\mathbf{\Lambda}t}\mathbf{V}$$

$$= \sum_{i=1}^{n} e^{\lambda_i t}\mathbf{u}_i\mathbf{v}_i^{\mathrm{T}} \implies R_i = \mathbf{u}_i\mathbf{v}_i^{\mathrm{T}}$$

then the participation values can be written as:

$$P_{kij} \triangleq \mathbf{u}_i^k\mathbf{v}_i^j$$
$$= \mathbf{e}_k^{\mathrm{T}} \underbrace{\mathbf{u}_i\mathbf{v}_i^{\mathrm{T}}}_{R_i} \mathbf{e}_j$$
$$\implies P_{kij} \triangleq \mathbf{e}_k^{\mathrm{T}} R_i \mathbf{e}_j. \qquad \square$$

The above definitions and theorems lead to a better understanding of the relationship between states and physical parameters.

4. Effect matrices

In the previous sections the following state-space representation has been derived:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{14}$$

where

$$\mathbf{A} = (\mathbf{J}_{SS} + \mathbf{J}_{SL}\mathbf{L}(\mathbf{I} - \mathbf{J}_{LL}\mathbf{L})^{-1}\mathbf{J}_{LS})\mathbf{S} \equiv \mathbf{J}\mathbf{S}. \tag{15}$$

Using this special form of the state-space equations, two equations based on eigenvalue sensitivity can be formed.

Lemma 4. *The partial derivative of an eigenvalue with respect to an energy storage element, s_j, when the structural state-space equation is used, is given by*

$$\frac{\partial \lambda_i}{\partial s_j} = \mathbf{v}_i^{\mathrm{T}}(\mathbf{J}t_j\mathbf{e}_j\mathbf{e}_j^{\mathrm{T}})\mathbf{u}_i \tag{16}$$

where t_j is a multiplication factor.

Proof.

$$\frac{\partial \lambda_i}{\partial q} = \mathbf{v}_i^{\mathrm{T}}\frac{\partial \mathbf{A}}{\partial q}\mathbf{u}_i$$

$$\frac{\partial \lambda_i}{\partial s_j} = \mathbf{v}_i^{\mathrm{T}}\frac{\partial \mathbf{A}}{\partial s_j}\mathbf{u}_i$$

$$= \mathbf{v}_i^{\mathrm{T}}\frac{\partial(\mathbf{J}_{SS} + \mathbf{J}_{SL}\mathbf{L}(\mathbf{I} - \mathbf{J}_{LL}\mathbf{L})^{-1}\mathbf{J}_{LS})\mathbf{S}}{\partial s_j}\mathbf{u}_i$$

or simply

$$\frac{\partial \lambda_i}{\partial s_j} = \mathbf{v}_i^{\mathrm{T}} \frac{\partial \mathbf{JS}}{\partial s_j} \mathbf{u}_i$$

as $\frac{\partial \mathbf{J}}{\partial s_j} = 0$ i.e., is constant with respect to s_j,

$$\frac{\partial \lambda_i}{\partial s_j} = \mathbf{v}_i^{\mathrm{T}} \mathbf{J} \frac{\partial \mathbf{S}}{\partial s_j} \mathbf{u}_i$$

and since $\mathbf{S}$ is diagonal, $\partial \mathbf{S}/\partial s_j = t_j \mathbf{e}_j \mathbf{e}_j^{\mathrm{T}}$, thus

$$\frac{\partial \lambda_i}{\partial s_j} = \mathbf{v}_i^{\mathrm{T}} \mathbf{J} t_j \mathbf{e}_j \mathbf{e}_j^{\mathrm{T}} \mathbf{u}_i$$

where s stands for energy storage elements as defined before.   □

It is apparent that the multiplication factor t_j can be calculated in a straightforward way. The multiplication factor depends on the form of the energy storage element as explained in Section 2. Table 1 gives a list of these multiplication factors.

Similarly, a second equation is formed as follows.

Lemma 5. *The partial derivative of an eigenvalue with respect to an energy dissipation element, l_j, when the structural state-space equation is used, is given by*

$$\frac{\partial \lambda_i}{\partial l_j} = \mathbf{v}_i^{\mathrm{T}} (\mathbf{J}_{SL} z_j \mathbf{e}_j \mathbf{e}_j^{\mathrm{T}} \mathbf{J}_{LS} \mathbf{S}) \mathbf{u}_i \tag{17}$$

where z_j is a multiplication factor.

Proof.

$$\frac{\partial \lambda_i}{\partial q} = \mathbf{v}_i^{\mathrm{T}} \frac{\partial \mathbf{A}}{\partial q} \mathbf{u}_i,$$

Table 1
A list of multiplication factors for effect matrices

Domain of interest	R element	$z_j = \frac{\partial R}{\partial l}$	I element	$t_j = \frac{\partial I}{\partial s}$	C element	$t_j = \frac{\partial C}{\partial s}$
Mechanical translation	b	1	$\frac{1}{m}$	$-\frac{1}{m^2}$	k	1
Mechanical rotation	c	1	$\frac{1}{J}$	$-\frac{1}{J^2}$	k	1
Hydraulic	R	1	$\frac{1}{I}$	$-\frac{1}{I^2}$	$\frac{1}{C}$	$-\frac{1}{C^2}$
Electrical	R	1	$\frac{1}{L}$	$-\frac{1}{L^2}$	$\frac{1}{C}$	$-\frac{1}{C^2}$

$$\frac{\partial \lambda_i}{\partial l_j} = \mathbf{v}_i^{\mathrm{T}} \frac{\partial \mathbf{A}}{\partial l_j} \mathbf{u}_i$$

$$= \mathbf{v}_i^{\mathrm{T}} \frac{\partial (\mathbf{J}_{SS} + \mathbf{J}_{SL}\mathbf{L}(\mathbf{I} - \mathbf{J}_{LL}\mathbf{L})^{-1}\mathbf{J}_{LS})\mathbf{S}}{\partial l_j} \mathbf{u}_i,$$

$$\frac{\partial \lambda_i}{\partial l_j} = \mathbf{v}_i^{\mathrm{T}} \left(\mathbf{J}_{SL} \frac{\partial \mathbf{L}}{\partial l_j} (\mathbf{I} - \mathbf{J}_{LL}\mathbf{L})^{-1}\mathbf{J}_{LS}\mathbf{S} + \mathbf{J}_{SL}\mathbf{L}(\mathbf{I} - \mathbf{J}_{LL}\mathbf{L})^{-2}\mathbf{J}_{LL}\mathbf{J}_{LS}\mathbf{S} \right) \mathbf{u}_i,$$

if $\mathbf{J}_{LL} = 0$ i.e., no causal connection between dissipation elements,

$$\frac{\partial \lambda_i}{\partial l_j} = \mathbf{v}_i^{\mathrm{T}} \mathbf{J}_{SL} \frac{\partial \mathbf{L}}{\partial l_j} \mathbf{J}_{LS}\mathbf{S}\mathbf{u}_i$$

and since $\mathbf{L}$ is diagonal, $\partial \mathbf{L}/\partial l_j = z_j \mathbf{e}_j \mathbf{e}_j^{\mathrm{T}}$, thus

$$\frac{\partial \lambda_i}{\partial l_j} = \mathbf{v}_i^{\mathrm{T}} \mathbf{J}_{SL} z_j \mathbf{e}_j \mathbf{e}_j^{\mathrm{T}} \mathbf{J}_{LS}\mathbf{S}\mathbf{u}_i$$

where l stands for energy dissipation elements as defined before. $\quad\square$

Here, for simplicity, it is assumed that $\mathbf{J}_{LL} \equiv 0$, i.e. none of the dissipation elements are directly casually related. This is not a critical assumption as this is a common case in structures. With this assumption $\mathbf{A}$ becomes as $\mathbf{A} = (\mathbf{J}_{SS} + \mathbf{J}_{SL}\mathbf{L}\mathbf{J}_{LS})\mathbf{S}$. Once again, a multiplication factor, z_j may be generated depending on the form of the dissipation element, see Table 1.

Using the eigenvalue sensitivities we can define two "effect" matrices, namely, one for energy storage, $\mathbf{E}_{IC}$, and one for energy dissipation elements, $\mathbf{E}_R$. The following six steps can be employed to form these matrices that will portray the relative contribution of physical elements on a selected eigenvalue:

(1) After forming the bond graph of the system, calculate the system matrices $\mathbf{S}$, $\mathbf{J}_{SS}$, $\mathbf{L}$, $\mathbf{J}_{SL}$, $\mathbf{J}_{LS}$, $\mathbf{J}_{LL}$ and $\mathbf{A}$.
(2) Form the state matrix of the system as $\mathbf{A} = \mathbf{JS}$ where $\mathbf{J}$ is defined as Eq. (15).
(3) Calculate the left and right eigenvector matrices of the state matrix $\mathbf{A}$, i.e. $\mathbf{V}$ and $\mathbf{U}$ matrices, respectively.
(4) For each eigenvalue calculate the following physical parameter sensitivities,

$$\frac{\partial \lambda_i}{\partial s_j} = \mathbf{v}_i^{\mathrm{T}} \mathbf{J} t_j \mathbf{e}_j \mathbf{e}_j^{\mathrm{T}} \mathbf{u}_i,$$

$$\frac{\partial \lambda_i}{\partial l_j} = \mathbf{v}_i^{\mathrm{T}} \left(\mathbf{J}_{SL} \frac{\partial \mathbf{L}}{\partial l_j} (\mathbf{I} - \mathbf{J}_{LL}\mathbf{L})^{-1}\mathbf{J}_{LS}\mathbf{S} + \mathbf{J}_{SL}\mathbf{L}(\mathbf{I} - \mathbf{J}_{LL}\mathbf{L})^{-2}\mathbf{J}_{LL}\mathbf{J}_{LS}\mathbf{S} \right) \mathbf{u}_i$$

where $i = 1, \dots, n$. Additionally $j = 1, \dots, r$ for energy storage elements, and $j = 1, \dots, m$ for dissipation elements.
(5) Take the absolute value of the results, i.e. calculate $|\partial \lambda_i/\partial s_j|$ and $|\partial \lambda_i/\partial l_j|$.
(6) Form the $\mathbf{E}_{IC}$ and $\mathbf{E}_R$ matrices using the eigenvalue sensitivity values such that each row corresponds to one eigenvalue, and each column corresponds to one

energy storage or energy dissipation element, respectively, i.e.

$$
\mathbf{E}_{IC} = \begin{bmatrix} \left|\frac{\partial \lambda_1}{\partial s_1}\right| & \left|\frac{\partial \lambda_1}{\partial s_2}\right| & \cdots & \left|\frac{\partial \lambda_1}{\partial s_r}\right| \\ \left|\frac{\partial \lambda_2}{\partial s_1}\right| & \left|\frac{\partial \lambda_2}{\partial s_2}\right| & \cdots & \left|\frac{\partial \lambda_2}{\partial s_r}\right| \\ \vdots & \vdots & \vdots & \vdots \\ \left|\frac{\partial \lambda_n}{\partial s_1}\right| & \left|\frac{\partial \lambda_n}{\partial s_2}\right| & \cdots & \left|\frac{\partial \lambda_n}{\partial s_r}\right| \end{bmatrix} \quad \text{and} \quad \mathbf{E}_R = \begin{bmatrix} \left|\frac{\partial \lambda_1}{\partial l_1}\right| & \left|\frac{\partial \lambda_1}{\partial l_2}\right| & \cdots & \left|\frac{\partial \lambda_1}{\partial l_m}\right| \\ \left|\frac{\partial \lambda_2}{\partial l_1}\right| & \left|\frac{\partial \lambda_2}{\partial l_2}\right| & \cdots & \left|\frac{\partial \lambda_2}{\partial l_m}\right| \\ \vdots & \vdots & \vdots & \vdots \\ \left|\frac{\partial \lambda_n}{\partial l_1}\right| & \left|\frac{\partial \lambda_n}{\partial l_2}\right| & \cdots & \left|\frac{\partial \lambda_n}{\partial l_m}\right| \end{bmatrix}.
$$

It is important to note that the introduction of the effect matrix, $\mathbf{E}_{IC}$, constitutes the superset of a method developed in [14]. Ye and Youcef-Toumi [14] outlined a procedure for the identification of components that are irrelevant to a given eigenvalue, in which both left and right eigenvectors of λ_i have to be considered. In contrast, the effect matrices not only identify the irrelevant components but also give a relative measure of their contribution to a selected eigenvalue. In the next section a sub-procedure for physical model reduction method based on decomposition procedures will be given. This sub-procedure uses the idea of Ye and Youcef-Toumi [14], also see Orbak et al. [7].

4.1. Physical model reduction

For complex dynamic systems, it is often useful to find a simplified model for purposes such as controller design, parameter optimization, design assessment under uncertainty, and to get better insight into the system behavior. In recent years, physical-based model reduction procedures have been developed. For example, in [5], Louca et al. propose a method that deal with the identification of components that have little influence on the response of the overall system. This method is established on an energy criteria, i.e. an energy-based "element activity index" is defined that is calculated as the ratio of the energy flowing through an element to the total system energy. Then the bonds of the bond graph that are deemed unnecessary are eliminated by removing the low activity elements according to a chosen appropriate threshold value.

Similarly, in [6], a physical-based model reduction procedure is developed and assessed. The method leads to an appropriate reduced-order model while again retaining a physical relevance to the full order model by indicating which subsystems to retain or remove in a systemic way. This proposed methodology in [6] exploits the concept of decomposition of physical systems suitable for the identification of dominant subsystems. Although this procedure is efficient, when a system has uniform parameters or has numerically identical loop gains, it may fail to identify all of the modes of the system. In such cases the following three step sub-procedure improves the results:

(1) First, the parameter that is irrelevant for a given mode is identified [14]. The given mode can be selected as one of the modes of the reduced-order model as identified by the decomposition procedures.

(2) The causal paths with this component and the rest of the system is examined on the bond graph.
(3) The irrelevant component and the components that have a causal relation to it are placed in a separate subsystem.

Once the subsystems are identified the rest of the physical domain model reduction technique can be applied without any change, i.e. the residue information is used to select the physical reduced-order model.

Although the above-improved procedure is very effective, one needs to observe exact "zero" components in the eigenvectors. Alternatively, the effect matrices introduced in this section can be utilized to obtain a more efficient procedure. The use of effect matrices adds additional flexibility to the step of obtaining subsystems, i.e. depending on the relationship using the bond graph causality assignment, the physical parameters that do not affect an eigenvalue of interest can be removed. Furthermore, the physical elements can be put in subsystems that define their specific behavior.

The physical domain model reduction procedure based on effect matrices is performed using the following five steps:

(1) Calculate the $\mathbf{E}_{IC}$ and $\mathbf{E}_R$ matrices.
(2) Calculate the residues of the system using $R_i = \mathbf{u}_i \mathbf{v}_i^{\mathrm{T}}$ and indicate the eigenvalues that are more important, i.e. have the most contribution to the response and thus will be retained in a reduced-order model.
(3) Identify relevant or most effective energy storage elements for the selected eigenvalue in Step 2.
(4) Identify the energy dissipation elements whose inputs are linear combinations of the outputs of the energy storage elements identified in Step 3.
(5) Collect the elements identified in Steps 3 and 4. These elements constitute subsystem that generates the selected behavior and thus provide the reduced-order model for the given system.

The identification of the energy dissipation elements of Step 4 is completed as follows:

- For each of the dissipation elements, follow the causal path initiated at its output, until all branches of the causal path reach an energy storage element.
- The input to the dissipation element is a linear combination of the identified energy storage elements, if every energy storage element reached by the branches of the causal path is the energy storage element identified in Step 3.

In the physical model reduction procedure the relevance of energy dissipation elements to eigenvalues is determined in Step 4. The proof of this statement is accomplished by adopting a theorem in [14].

Theorem 6. *A dissipation element has the greatest contribution on the ith eigenvalue λ_i, if its causal input is a linear combination of the outputs of the energy storage elements that have the highest contribution on λ_i.*

Proof. Assume that a dissipation element R_j with the parameter l_j is given. As

$$\dot{\mathbf{x}} = \mathbf{J}_{SS}\mathbf{z} + \mathbf{J}_{SL}\mathbf{d}_{\text{out}} + \mathbf{J}_{SU}\mathbf{u}$$

the parameter of the dissipation element enters the state matrix $\mathbf{A}$ via the output of the element $\mathbf{d}_{\text{out}}$. If the causal input of R_j is a linear combination of the energy storage elements, one has,

$$\mathbf{d}_{\text{out}_j} = l_j \cdot \sum_{k=r+1}^{n} \alpha_k s_k \mathbf{x}_k \tag{18}$$

where α_k are real numbers. Now suppose λ_i changes when l_j is set to be βl_j where β is a real number not equal to unity, i.e. $\beta \neq 1$. This means that λ_i changes when

$$\mathbf{d}_{\text{out}_j} = (\beta l_j) \cdot \sum_{k=r+1}^{n} \alpha_k s_k \mathbf{x}_k$$

$$= l_j \cdot \sum_{k=r+1}^{n} \alpha_k (\beta s_k) \mathbf{x}_k. \tag{19}$$

This means that changing l_j to βl_j while keeping all s_k constant is equivalent to changing all s_k's ($k \in [r+1, n]$) to βs_k while keeping l_j constant. Thus, if λ_i changes when l_j changes, it must be true that λ_i changes when s_k's change. Therefore it can be concluded that:

- The dissipation elements that have larger numerical values in effect matrix $\mathbf{E}_R$ and have causal connection to the energy storage elements with larger numerical values in effect matrix $\mathbf{E}_{IC}$ contribute more to λ_i.
- The dissipation elements that have zero values in effect matrix $\mathbf{E}_R$ and has causal connection to the energy storage elements with zero values in effect matrix $\mathbf{E}_{IC}$ are irrelevant to λ_i [14]. □

It is important to note that in this method, the effect of components on a selected eigenvalue is observed, i.e. the effect matrix is given for a specified eigenvalue. On contrast, the methods of Louca et al. [5] identify the relevant or irrelevant components for a specified frequency range of interest.

5. Implementation examples

In this section three examples will be given: The first two examples illustrate the construction of effect matrices for obtaining the relationship between eigenvalues and physical parameters, and the last one shows the use of the procedure for model reduction. The first example is a multi mass–spring–damper system. The second is a standard bond graph example that can easily be found in electrical or mechanical systems, which has repeated eigenvalues. The example chosen for the repeated eigenvalue case is taken and adopted from [14], whereas the example for the model reduction sub-procedure is taken and adopted from [6].

5.1. A mass–spring–damper system

In this section, the eigenvalue sensitivity method will be applied to a mass–spring–damper system shown in Fig. 1. The bond graph representation of this system is seen in Fig. 2. This system is of order six, but with the states chosen in bond graph representation it possesses three excess states. The numerical values of the physical parameters are given as: $m_1 = m_2 = m_3 = 1\,\text{kg}$, $k_1 = 4\,\text{N/m}$, $k_2 = k_3 = k_4 = 1\,\text{N/m}$, $k_5 = 2.5\,\text{N/m}$, $k_6 = 5\,\text{N/m}$, and $b_1 = 1.5\,\text{N s/m}$, $b_2 = b_3 = 0.5\,\text{N s/m}$, $b_4 = 1\,\text{N s/m}$, $b_5 = 0.4\,\text{N s/m}$, $b_6 = 0.2\,\text{N s/m}$. For this system, the following matrices can be constructed:

$$\mathbf{S} = \begin{bmatrix} \frac{1}{m_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{m_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{m_3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & k_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & k_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & k_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & k_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & k_6 \end{bmatrix}, \tag{20}$$
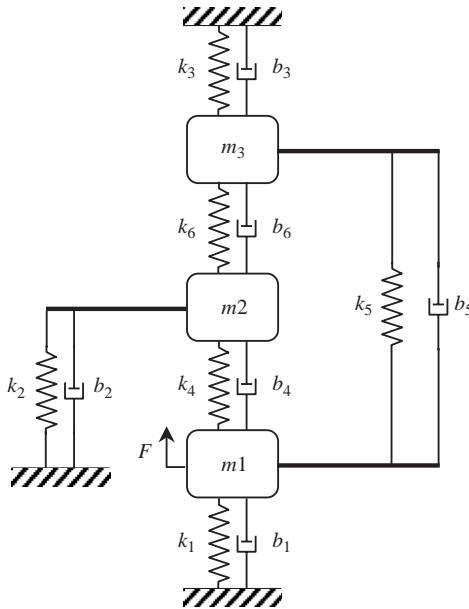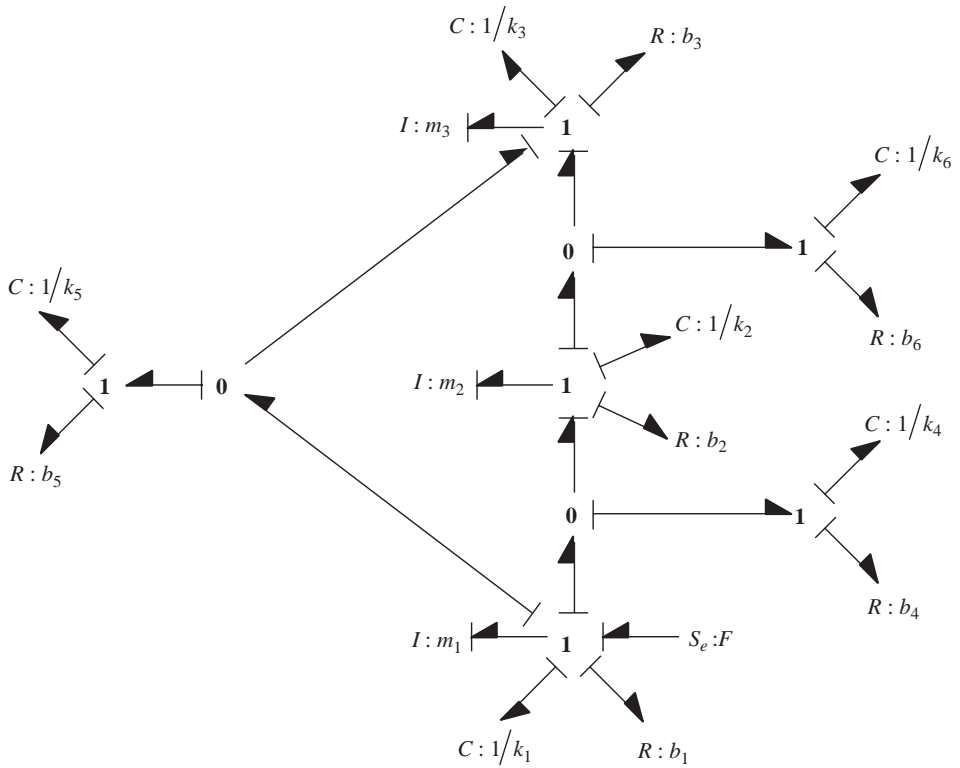

Fig. 1. A mass–spring–damper system.

Fig. 2. Bond graph representation of the mass–spring–damper system.

$$\mathbf{J}_{SS} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad (21)$$

$$\mathbf{J}_{SL} = \begin{bmatrix} -1 & 0 & 0 & -1 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{22}$$

$$\mathbf{L} = \mathrm{diag}[\, b_1 \quad b_2 \quad b_3 \quad b_4 \quad b_5 \quad b_6 \,], \quad \mathbf{J}_{LL} = \mathbf{0}_{6 \times 6}, \tag{23}$$

$$\mathbf{J}_{LS} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{24}$$

With these matrices the dynamic $\mathbf{A}$ matrix of the system is obtained as

$$\mathbf{A} = \begin{bmatrix} -\frac{(b_1+b_4+b_5)}{m_1} & \frac{b_4}{m_2} & \frac{b_5}{m_3} & -k_1 & 0 & 0 & -k_4 & -k_5 & 0 \\ \frac{b_4}{m_1} & -\frac{(b_2+b_4+b_6)}{m_2} & \frac{b_6}{m_3} & 0 & -k_2 & 0 & k_4 & 0 & -k_6 \\ \frac{b_5}{m_1} & \frac{b_6}{m_2} & -\frac{(b_3+b_5+b_6)}{m_3} & 0 & 0 & -k_3 & 0 & k_5 & k_6 \\ \frac{1}{m_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{m_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{m_3} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{m_1} & -\frac{1}{m_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{m_1} & 0 & -\frac{1}{m_3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{m_2} & -\frac{1}{m_3} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{25}$$

The eigenvalues of matrix $\mathbf{A}$ are calculated to be:

$$\lambda_{1,2} = -0.7292 \pm 3.5448i,$$
$$\lambda_{3,4} = -1.7433 \pm 2.2674i,$$
$$\lambda_{5,6} = -0.3774 \pm 1.2456i.$$

Additionally, with the calculation of eigenvectors, the effect matrices are formed as

$$\mathbf{E}_{IC} = \begin{bmatrix} 0.1005 & \mathbf{0.6548} & \mathbf{1.0945} & 0.0077 & 0.0500 & 0.0836 & 0.0185 & 0.1416 & \mathbf{0.2626} \\ 0.1005 & \mathbf{0.6548} & \mathbf{1.0945} & 0.0077 & 0.0500 & 0.0836 & 0.0185 & 0.1416 & \mathbf{0.2626} \\ 1.3926 & 0.4001 & 0.0176 & 0.1702 & 0.0489 & 0.0022 & 0.4009 & 0.2091 & 0.0309 \\ 1.3926 & 0.4001 & 0.0176 & 0.1702 & 0.0489 & 0.0022 & 0.4009 & 0.2091 & 0.0309 \\ 0.1064 & 0.3048 & 0.2720 & 0.0628 & 0.1799 & 0.1606 & 0.0308 & 0.0227 & 0.0008 \\ 0.1064 & 0.3048 & 0.2720 & 0.0628 & 0.1799 & 0.1606 & 0.0308 & 0.0227 & 0.0008 \end{bmatrix} \tag{26}$$

and

$$\mathbf{E}_R = \begin{bmatrix} 0.0278 & 0.1809 & 0.3024 & 0.0671 & 0.5125 & \mathbf{0.9504} \\ 0.0278 & 0.1809 & 0.3024 & 0.0671 & 0.5125 & \mathbf{0.9504} \\ 0.4869 & 0.1399 & 0.0062 & 1.1466 & 0.5980 & 0.0885 \\ 0.4869 & 0.1399 & 0.0062 & 1.1466 & 0.5980 & 0.0885 \\ 0.0818 & 0.2342 & 0.2090 & 0.0400 & 0.0295 & 0.0010 \\ 0.0818 & 0.2342 & 0.2090 & 0.0400 & 0.0295 & 0.0010 \end{bmatrix}. \tag{27}$$

Here, in matrix $\mathbf{E}_{IC}$ each column corresponds to one energy storage element, in the same order as that of matrix $\mathbf{S}$, and each row corresponds to one eigenvalue. For example the values $\mathbf{E}_{IC_{31}}$ and $\mathbf{E}_{IC_{41}}$ of 1.3926 give the relative weights of m_1 on eigenvalues $\lambda_{3,4} = -1.7433 \pm 2.2674i$. Similarly, in matrix $\mathbf{E}_R$ each column corresponds to one energy dissipation element in the same order of matrix $\mathbf{L}$, and each row corresponds to one eigenvalue. From the effect matrices, one can observe that the parameters m_2, m_3, k_6 and b_6 (shown in bold case) have the most effect on $\lambda_{1,2}$ values. This result indicates that these physical parameters influence that eigenvalue. As these parameters are directly casually related in the bond graph, they can be put in a subsystem that portrays the behavior of the first complex-conjugate eigenvalue. This result is important, as it directly helps in identifying the relevant parameters for a selected eigenvalue, and also allows the user to select the parameters to obtain a predefined eigenvalue.

5.2. A simple example with repeated roots

Consider the system given by the bond graph in Fig. 3 [14]. All parameter values except α are shown on the figure. For this example $\alpha = 1$ is chosen.
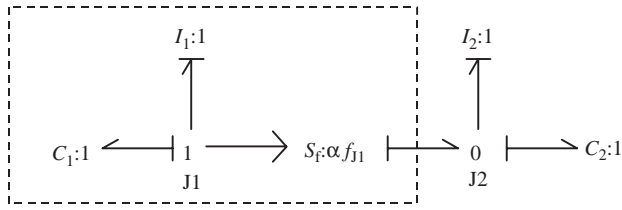
Fig. 3. A simple system.

For this system, using the same approach as in the first example the following system matrices can be constructed:

$$\mathbf{S} = \begin{bmatrix} \frac{1}{I_1} & 0 & 0 & 0 \\ 0 & \frac{1}{C_1} & 0 & 0 \\ 0 & 0 & \frac{1}{I_2} & 0 \\ 0 & 0 & 0 & \frac{1}{C_2} \end{bmatrix}, \quad \mathbf{J} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}. \tag{28}$$

With these two matrices the dynamic $\mathbf{A}$ matrix of the system is obtained as

$$\mathbf{A} = \mathbf{JS} = \begin{bmatrix} 0 & -\frac{1}{C_1} & 0 & 0 \\ \frac{1}{I_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{C_2} \\ \frac{1}{I_1} & 0 & -\frac{1}{I_2} & 0 \end{bmatrix}. \tag{29}$$

This system produces the symbolic eigenvalues as: $\pm\sqrt{\frac{1}{I_1 C_1}}i$, and $\pm\sqrt{\frac{1}{I_2 C_2}}i$. With the chosen parameter values the numerical eigenvalues are calculated as: $\pm i$, $\pm i$, which indicates that there are two repeated roots. Thus, generalized eigenvectors have to be calculated. As a result, the right and left eigenvector matrices for this $\mathbf{A}$ matrix are computed as

$$\mathbf{U} = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3 \quad \mathbf{u}_4], \tag{30}$$
$$\mathbf{V} = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{v}_4] \tag{31}$$

where

$$\mathbf{u}_1 = \begin{bmatrix} 0 \\ 0 \\ -0.2500i \\ 0.2500 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0.5000 \\ -0.5000i \\ 0 \\ -0.2500i \end{bmatrix},$$

$$\mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 0.2500i \\ 0.2500 \end{bmatrix}, \quad \mathbf{u}_4 = \begin{bmatrix} 0.5000 \\ 0.5000i \\ 0 \\ 0.2500i \end{bmatrix}$$

and

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ -1.0000 \\ -2.0000\mathrm{i} \\ 2.0000 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1.0000 \\ -1.0000\mathrm{i} \\ 0 \\ 0 \end{bmatrix},$$

$$\mathbf{v}_3 = \begin{bmatrix} 0 \\ 1.0000 \\ 2.0000\mathrm{i} \\ 2.0000 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 1.0000 \\ 1.0000\mathrm{i} \\ 0 \\ 0 \end{bmatrix}.$$

As this system has repeated eigenvalues the eigenvectors are the generalized eigenvectors. But as explained before, this does not alter the given derivations. Since there are no dissipation elements in this system, $\mathbf{E}_R = \mathbf{0}$, and $\mathbf{E}_{IC}$ is calculated to be

$$\mathbf{E}_{IC} = \begin{bmatrix} 0 & 0 & 0.5000 & 0.5000 \\ 0.5000 & 0.5000 & 0 & 0 \\ 0 & 0 & 0.5000 & 0.5000 \\ 0.5000 & 0.5000 & 0 & 0 \end{bmatrix}. \tag{32}$$

For this system the eigenvalues are in order of i, i, −i, −i. The effect matrix indicates that all the elements have the same effect on eigenvalues. It can be observed that these results are expected. It can also be observed that only $I_1 - C_1$ affect one set of eigenvalues, and $I_2 - C_2$ affect the other. This is consistent with the symbolic calculation.

5.3. A SISO physical example

In this subsection, the use of the sub-procedure for model reduction for a single-input single-output (SISO) physical system is presented.

Consider the system shown in Fig. 4. The bond graph representation of this system is displayed in Fig. 5.

Let's assume that the system has uniform parameters, i.e. $m_1 = m_2 = 1\,\mathrm{kg}$, $k_1 = k_2 = k_3 = 2\,\mathrm{N/m}$, $b_1 = b_2 = b_3 = 1\,\mathrm{N\,s/m}$. For this set of parameters the physical model reduction method described [6] is not efficient. This is because the local damping ratios and loop gains are numerically exactly the same, and the sum of loop gains are approximately the same. Thus, for model reduction the sub-procedure explained in this section will be applied.

For this system, the following system matrices can be constructed:

$$\mathbf{S} = \begin{bmatrix} \frac{1}{m_1} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{m_2} & 0 & 0 & 0 \\ 0 & 0 & k_1 & 0 & 0 \\ 0 & 0 & 0 & k_2 & 0 \\ 0 & 0 & 0 & 0 & k_3 \end{bmatrix}, \tag{33}$$

$$\mathbf{J}_{SS} = \begin{bmatrix} 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \tag{34}$$
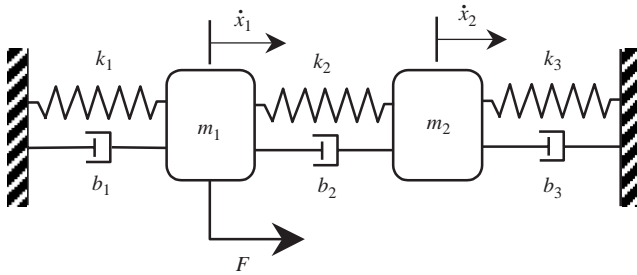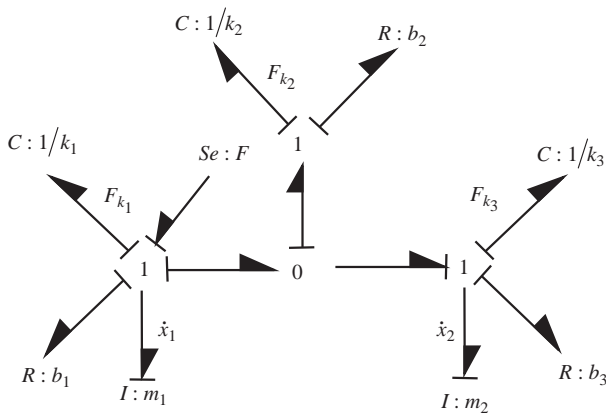

Fig. 4. A SISO physical system.

Fig. 5. Bond graph representation of the SISO physical system.

$$\mathbf{L} = \begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{bmatrix}, \quad \mathbf{J}_{LL} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{35}$$

$$\mathbf{J}_{SL} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{J}_{LS} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}. \tag{36}$$

With these matrices the dynamic $\mathbf{A}$ matrix of the system is obtained as

$$\mathbf{A} = \begin{bmatrix} -\frac{b_1+b_2}{m_1} & \frac{b_2}{m_2} & -k_1 & -k_2 & 0 \\ \frac{b_2}{m_1} & -\frac{b_2+b_3}{m_2} & 0 & k_2 & -k_3 \\ \frac{1}{m_1} & 0 & 0 & 0 & 0 \\ \frac{1}{m_1} & -\frac{1}{m_2} & 0 & 0 & 0 \\ 0 & \frac{1}{m_2} & 0 & 0 & 0 \end{bmatrix}. \tag{37}$$

The eigenvalues, and the right and left eigenvector matrices of matrix $\mathbf{A}$ are computed as

$$\lambda_{1,2} = -1.5000 \pm 1.9365i,$$
$$\lambda_{3,4} = -0.5000 \pm 1.3229i,$$
$$\lambda_5 = 0,$$

$$\mathbf{U} = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3 \quad \mathbf{u}_4 \quad \mathbf{u}_5], \tag{38}$$

$$\mathbf{V} = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{v}_4 \quad \mathbf{v}_5], \tag{39}$$

where

$$\mathbf{u}_1 = \begin{bmatrix} -0.4193 + 0.3969i \\ 0.4193 - 0.3969i \\ 0.2329 + 0.0361i \\ 0.4658 + 0.0722i \\ -0.2329 - 0.0361i \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -0.4193 - 0.3969i \\ 0.4193 + 0.3969i \\ 0.2329 - 0.0361i \\ 0.4658 - 0.0722i \\ -0.2329 + 0.0361i \end{bmatrix},$$

$$\mathbf{u}_3 = \begin{bmatrix} 0.3906 + 0.4252i \\ 0.3906 + 0.4252i \\ 0.1836 - 0.3646i \\ 0 \\ 0.1836 - 0.3646i \end{bmatrix}, \quad \mathbf{u}_4 = \begin{bmatrix} 0.3906 - 0.4252i \\ 0.3906 - 0.4252i \\ 0.1836 + 0.3646i \\ 0 \\ 0.1836 + 0.3646i \end{bmatrix}, \quad \mathbf{u}_5 = \begin{bmatrix} 0 \\ 0 \\ -0.5774 \\ 0.5774 \\ 0.5774 \end{bmatrix}$$

and

$$\mathbf{v}_1 = \begin{bmatrix} -0.0838 + 0.5413i \\ 0.0838 - 0.5413i \\ 0.3075 + 0.3248i \\ 0.6149 + 0.6495i \\ -0.3075 - 0.3248i \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -0.0838 - 0.5413i \\ 0.0838 + 0.5413i \\ 0.3075 - 0.3248i \\ 0.6149 - 0.6495i \\ -0.3075 + 0.3248i \end{bmatrix},$$

$$\mathbf{v}_3 = \begin{bmatrix} 0.4135 + 0.2082i \\ 0.4135 + 0.2082i \\ 0.4821 - 0.4429i \\ 0 \\ 0.4821 - 0.4429i \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 0.4135 - 0.2082i \\ 0.4135 - 0.2082i \\ 0.4821 + 0.4429i \\ 0 \\ 0.4821 + 0.4429i \end{bmatrix}, \quad \mathbf{v}_5 = \begin{bmatrix} 0 \\ 0 \\ -0.5774 \\ 0.5774 \\ 0.5774 \end{bmatrix}.$$

Additionally, the effect matrices are calculated as

$$\mathbf{E}_{IC} = \begin{bmatrix} 0.7746 & 0.7746 & 0.1291 & 0.5164 & 0.1291 \\ 0.7746 & 0.7746 & 0.1291 & 0.5164 & 0.1291 \\ 0.3780 & 0.3780 & 0.1890 & \mathbf{0.0000} & 0.1890 \\ 0.3780 & 0.3780 & 0.1890 & \mathbf{0.0000} & 0.1890 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix} \qquad (40)$$

and

$$\mathbf{E}_R = \begin{bmatrix} 0.3162 & 1.2649 & 0.3162 \\ 0.3162 & 1.2649 & 0.3162 \\ 0.2673 & \mathbf{0.0000} & 0.2673 \\ 0.2673 & \mathbf{0.0000} & 0.2673 \\ 0.0000 & 0.0000 & 0.0000 \end{bmatrix}. \qquad (41)$$

Upon examination of the effect matrices, one can immediately see that the physical parameters k_2 and b_2 has no influence on $\lambda_{3,4} = -0.5000 \pm 1.3229i$ as their values are exactly zero. These values are shown in bold case in the matrices. The application of the sub-procedure outlined in Section 4.A. produces the two subsystems of Fig. 6. It is also noted that the effect matrices indicate the symmetry of the system.
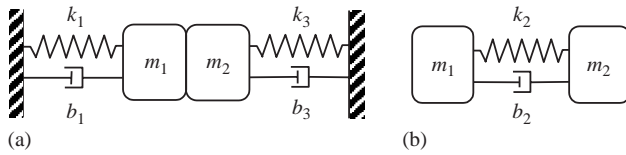
Fig. 6. Subsystems of the SISO physical system: (a) Subsystem 1, (b) Subsystem 2.
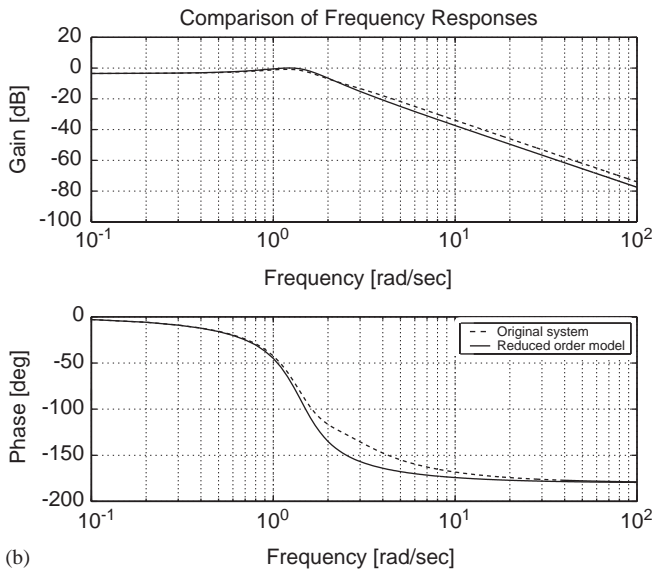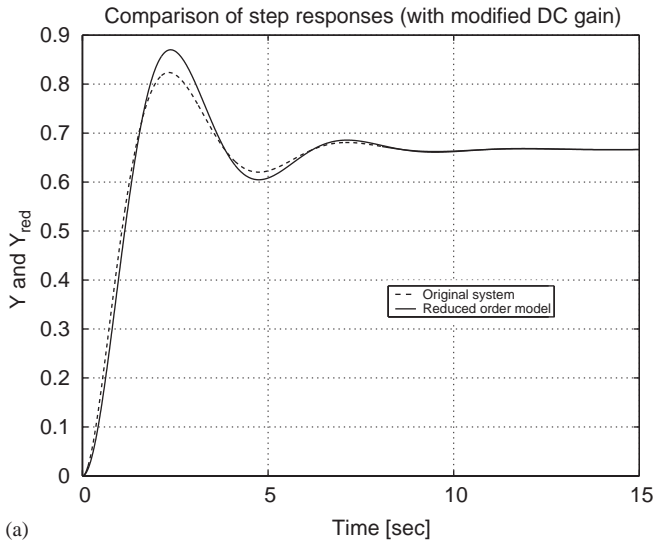

Fig. 7. Comparison of full- and reduced-order model with uniform parameters: (a) Comparison of step responses, (b) Comparison of Bode plots.

From this point, if the physical domain model reduction given in [6] is worked out as usual, the simulated results are as shown in Fig. 7. In this type of physical model reduction procedures there can be a DC gain discrepancy between the reduced and the full-order models that can be corrected easily. The DC gain difference may occur in cases where some parameters are eliminated without compensating their effects on the system as described previously. As it can be seen from these figures, the results are acceptable and the reduced-order model provides a good approximation with this modification of gain.

6. Conclusions

In this paper, the identification of parameters that influence a given eigenvalue of the overall system has been explored and the effect matrix method has been introduced. For this purpose, a set of theorems and definitions are proposed that lead to an efficient procedure for the identification of parameters that determine a given eigenvalue of the overall system. In this approach, a special type of state-space description obtained from bond graphs is utilized. After the calculation of eigenvectors and the defined effect matrices, the relative importance of physical parameters for a selected eigenvalue is readily obtained. Furthermore two physical-based model reduction procedures are explained: a sub-procedure is given to improve the decomposition based physical model reduction method that is found in the literature, and one new procedure for physical model reduction is given that utilizes the effect matrices. Three examples are given to illustrate the results.

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