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POSITIVE REAL DECENTRALIZED CONTROLLER SYNTHESIS FOR VIBRATION CONTROL

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ABSTRACT

In this paper, a linear matrix inequality (LMI) approach is presented for synthesis of full/reduced order H_∞ vibration controllers. Being decentralized and strictly positive realness are given constraints on the controller and closed loop transfer functions, respectively. All criteria and their interaction between each other are clearly explained in terms of linear matrix inequalities (LMIs). In order to find a common solution to these inequalities, an alternating projection algorithm combined with semidefinite programming (SDP) is used. An example is presented to demonstrate the approach.

INTRODUCTION

Vibration control is an important issue for harmonically excited structures. The main goal of control design is to reduce peak amplitude values of steady state oscillations; characteristics describing the intensity of vibration which have considerably important effects on the performance and safety of the system. For that purpose, a considerable amount of work has been devoted to the vibration control (Öztürk, 1997; Chilali and Gahinet, 1996; Gahinet and Apkarian, 1994). Controller design is an iterative process between the controller parameters and performance specifications of the controller inserted feedback system. In the controller synthesis, one must adjust the controller parameters until the performance of the whole system is satisfactory. In this paper; a linear matrix inequality approach to the design of vibration absorbers was developed. To determine the unknown parameters of desired controller, the bounded real lemma for the closed loop system was considered as a LMI(linear matrix inequality) problem. Being decentralized and positive realness are considered to be constraints on the controller transfer function.

One of possible controller structures is decentralized controller structure, where the main idea is the input/output pairing. In this structure; the controller transfer function matrix is in a diagonal or block diagonal form. Each input, or set of inputs, are assigned to the control of one particular output, or set of outputs. So the outputs can be easily controlled. The existence of positive realness in the controller structure is the only way to realize a controller by combination of passive elements, such as masses, springs and dampers. Controllers may be passive or active. Passive controllers have an energy-dissipation property (Xie, et. al. 1998). Linear matrix inequalities are the tools to cast the constraints on controller parameters with the expected performance, stability of the system into a matrix form, that in turn prepares a basis for the efficient numerical computation of the con-

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troller parameters (Dullerud and Paganini, 2000; Ghaoui and Niculescu, 1997; Vanantwerp and Braatz, 2000). After the controller parameters are obtained, various model reduction techniques can be applied to result in the most efficient simple-structured vibration controller.

PROBLEM FORMULATION

Let the n_p^{th} order linear time-invariant generalized vibrating plant be described by the state space equations:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w(t) + \sum_{i=1}^{N} B_{2i} u_i(t), \\ z(t) &= C_1 x(t) + D_{11} w(t) + \sum_{i=1}^{N} D_{12i} u_i(t), \\ y_i(t) &= C_{2i} x(t) + D_{21i} w(t) + D_{22i} u_i(t), \end{aligned} \tag{1}$$

for $i = 1, .., N$, total number of controller forces acting on the plant.

The generalized plant P contains what is usually called the vibrating plant in a vibration control problem plus all frequency-dependent weighting functions and $x(t) \in R^{n_p}$ is the state vector of the system. The disturbance vector $w(t) \in R^{n_w}$ contains all external inputs, including disturbances, sensor noise, and commands. $z(t) \in R^{n_z}$ is the vibration amplitude vector to be minimized, whereas $y_i(t) \in R^{n_y}$ and $u_i(t) \in R^{n_u}$ are the ith observation vector representing the measured variables,here velocities, and corresponding ith control input vector, respectively. The matrices $A, B_{11}, B_{12}, C_{11}, D_{11}, D_{12}, C_{21}, D_{21}, D_{22}$ are constant and compatible in dimension with corresponding vectors.

Arranging the matrices in the form;

$$B_2 = \begin{bmatrix} B_{21} \; B_{22} \cdots \; B_{2N} \end{bmatrix}$$

$$D_{12} = \begin{bmatrix} D_{121} \; D_{122} \cdots \; D_{12N} \end{bmatrix}$$

$$C_2 = \begin{bmatrix} C_{21} \\ C_{22} \\ \vdots \\ C_{2N} \end{bmatrix}$$

$$D_{21} = \begin{bmatrix} D_{211} \\ D_{212} \\ \vdots \\ D_{21N} \end{bmatrix}$$

$$D_{22} = \begin{bmatrix} D_{221} \; D_{222} \cdots \; D_{22N} \end{bmatrix}$$

One can obtain the plant transfer function in state space form as

$$\begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix} \tag{2}$$

or in s-domain as

$$\begin{bmatrix} z \\ y \end{bmatrix} = P(s) \begin{bmatrix} w \\ u \end{bmatrix} = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{11}(s) & P_{12}(s) \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \tag{3}$$

where

$$P_{ij}(s) = C_i(sI - A)^{-1} B_j + D_{ij}$$

with $i, j = 1, 2$.

On the other hand, the dynamic equations of linear time-invariant controllers of fixed order n_c are given as:

$$\begin{aligned} \dot{x}_c(t) &= A_c x_c(t) + B_c y(t) \\ u(t) &= C_c x_c(t) + D_c y(t) \end{aligned} \tag{4}$$

and in the matrix form; it becomes

$$\begin{bmatrix} \dot{x}_c \\ u \end{bmatrix} = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \begin{bmatrix} x_c \\ y \end{bmatrix} \tag{5}$$

where $x_c \in R^{n_c}$ is the controller state.

Here the controller transfer function matrix is

$$K = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \tag{6}$$

DECENTRALIZED CONTROLLER

For a decentralized controller with N-controller force action on the plant; the matrices A_c, B_c, C_c, D_c, consist of N sub-matrices $\hat{A}_i, \hat{B}_i, \hat{C}_i, \hat{D}_i$, in following form:

$$\begin{aligned} A_c &= diag([\hat{A}_1]_{\hat{n}_1 \times \hat{n}_1}, [\hat{A}_2]_{\hat{n}_2 \times \hat{n}_2}, \cdots, [\hat{A}_N]_{\hat{n}_N \times \hat{n}_N})_{n_c \times n_c} \\ B_c &= diag([\hat{B}_1]_{\hat{n}_1 \times 1}, [\hat{B}_2]_{\hat{n}_2 \times 1}, \cdots, [\hat{B}_N]_{\hat{n}_N \times 1})_{n_c \times N} \\ C_c &= diag([\hat{C}_1]_{1 \times \hat{n}_1}, [\hat{C}_2]_{1 \times \hat{n}_2}, \cdots, [\hat{C}_N]_{1 \times \hat{n}_N})_{N \times n_c} \\ D_c &= diag([\hat{D}_1]_{1 \times 1}, [\hat{D}_2]_{1 \times 1}, \cdots, [\hat{D}_N]_{1 \times 1})_{N \times N} \end{aligned} \tag{7}$$

with

$$n_c = \sum_{i=1}^{N} \hat{n}_i \tag{8}$$

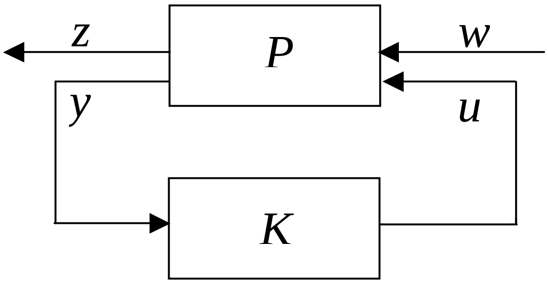

Figure 1. Generalized plant-controller con£guration

The transfer function of a decentralized controller in s-domain is given as

$$K(s) = C_c(sI - A_c)^{-1}B_c + D_c \tag{9}$$

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix} = K(s) \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}$$

$$= \begin{bmatrix} \hat{K}_1(s) & 0 & \cdots & 0 \\ 0 & \hat{K}_2(s) & \cdots & 0 \\ \cdots & 0 & \hat{K}_3(s) & 0 \\ 0 & \cdots & 0 & \hat{K}_N(s) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} \tag{10}$$

where

$$\hat{K}_i(s) = \hat{C}_i(sI - \hat{A}_i)^{-1}\hat{B}_i + \hat{D}_i$$

CLOSED LOOP TRANSFER FUNCTION

For a linear controller with transfer function $K(s)$ connected from y to u, the closed-loop transfer function from w to z can be written as a linear fractional transformation (Figure 1)in s-domain:

$$\begin{aligned} T_{zw}(s) &= F_l(P, K) \\ &= P_{11}(s) + P_{12}(s)K(s)(I - P_{22}(s)K(s))^{-1}P_{21}(s) \end{aligned} \tag{11}$$

or in state space form:

$$\left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right] \tag{12}$$

where

$$\bar{A} = \begin{bmatrix} A + B_2D_c(I - D_{22}D_c)^{-1}C_2 & B_2[C_c + D_c(I - D_{22}D_c)^{-1}D_{22}C_c \\ B_c(I - D_{22}D_c)^{-1}C_2 & A_c + B_c(I - D_{22}D_c)^{-1}D_{22}C_c] \end{bmatrix}$$

$$\bar{B} = \begin{bmatrix} B_1 + B_2D_c(I - D_{22}D_c)^{-1}D_{21} \\ B_c(I - D_{22}D_c)^{-1}D_{21} \end{bmatrix}$$

$$\bar{C} = \begin{bmatrix} C_1 + D_{12}D_c(I - D_{22}D_c)^{-1}C_2 & D_{12}[C_c + D_c(I - D_{22}D_c)^{-1}D_{22}C_c] \end{bmatrix}$$

$$\bar{D} = D_{11} + D_{12}D_c(I - D_{22}D_c)^{-1}D_{21}$$

If the open-loop system is augmented with the states corresponding to the controller, the following augmented system can be obtained:

$$\begin{bmatrix} \dot{x} \\ \dot{x}_c \\ \hline z \\ \hline x_c \\ y \end{bmatrix} = \left[\begin{array}{cc|c|cc} A & 0 & B_1 & 0 & B_2 \\ 0 & 0 & 0 & I_{n_c} & 0 \\ \hline C_1 & 0 & D_{11} & 0 & D_{12} \\ \hline 0 & I_{n_c} & 0 & 0 & 0 \\ C_2 & 0 & D_{21} & 0 & D_{22} \end{array}\right] \begin{bmatrix} x \\ x_c \\ \hline w \\ \hline \dot{x}_c \\ u \end{bmatrix} \tag{13}$$

equivalently,

$$\begin{bmatrix} \dot{\tilde{x}} \\ z \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} \tilde{A} & \tilde{B}_1 & \tilde{B}_2 \\ \tilde{C}_1 & \tilde{D}_{11} & \tilde{D}_{12} \\ \tilde{C}_2 & \tilde{D}_{21} & \tilde{D}_{22} \end{bmatrix} \begin{bmatrix} \tilde{x} \\ w \\ \tilde{u} \end{bmatrix}$$

where

$$\tilde{x} = \begin{bmatrix} x \\ x_c \end{bmatrix}, \tilde{y} = \begin{bmatrix} x_c \\ y \end{bmatrix}, \tilde{u} = \begin{bmatrix} \dot{x}_c \\ u \end{bmatrix}$$

with

$$\tilde{u} = K\tilde{y}$$

$$\begin{bmatrix} \dot{\tilde{x}} \\ z \end{bmatrix} = \begin{bmatrix} \bar{A} & \bar{B}_1 \\ \bar{C}_1 & \bar{D}_{11} \end{bmatrix} \begin{bmatrix} \tilde{x} \\ w \end{bmatrix}.$$

The closed-loop system matrix can be written as an affine function of the controller matrix as follows:

$$\begin{bmatrix} \bar{A} & \bar{B}_1 \\ \bar{C}_1 & \bar{D}_{11} \end{bmatrix} = \begin{bmatrix} \tilde{A} & \tilde{B}_1 \\ \tilde{C}_1 & \tilde{D}_{11} \end{bmatrix} + \begin{bmatrix} \tilde{B}_2 \\ \tilde{D}_{12} \end{bmatrix} \check{K} \begin{bmatrix} \tilde{C}_2 & \tilde{D}_{21} \end{bmatrix} \tag{14}$$

where

$$\check{K} = \begin{bmatrix} A_c + B_c(I - D_{22}D_c)^{-1}D_{22}C_c & B_c(I - D_{22}D_c)^{-1} \\ C_c(I - D_{22}D_c)^{-1} & D_c(I - D_{22}D_c)^{-1} \end{bmatrix} \tag{15}$$

STRICTLY POSITIVE REALNESS

A system in which state variables and the output take nonnegative values whenever initial states and inputs are nonnegative is called a positive system. A property for linear systems subject to perturbations is passivity. A linear system is said to be passive if

$$\int_0^{\tau} u(t)^T y(t) dt \geq 0$$

for all u and $\tau \geq 0$.

Lemma 1. ***Positive Real Lemma*** *The passivity property for strictly positive realness of the closed loop system is equivalent to the existence of $Q = Q^T > 0$ such that*

$$\begin{bmatrix} \bar{A}^T Q + Q\bar{A} & Q\bar{B}_1 - \bar{C}_1^T \\ \bar{B}_1^T Q - \bar{C}_1 & -(\bar{D}_{11}^T + \bar{D}_{11}) \end{bmatrix} < 0. \quad (16)$$

in s-domain; the closed loop transfer function $T_{zw}(s)$ is strictly positive real if $T_{zw}(s)$ is asymptotically stable and $1/2(K(j\omega)^* + K(j\omega)) > 0 \ \forall \omega \in \Re$. Asymptotically stable transfer functions are transfer functions whose poles are in the open left half plane.

H_∞ CONTROL

$\|T_{zw}\|_\infty$ denotes H_∞ norm of the closed-loop transfer function T_{zw}, i.e. its largest gain across frequency in the singular value norm. The H_∞ norm measures the system input-output gain for finite energy or finite RMS input signals. $\|T_{zw}\|_\infty = \sup_{w \in L_2, w \neq 0} (\|z\|_2 / \|w\|_2)$ with the constraint $\|T_{zw}\|_\infty < \gamma$ can be interpreted as a disturbance rejection performance.

Lemma 2. ***Bounded Real Lemma*** *Given a system of the form*

$$\begin{bmatrix} \dot{\tilde{x}} \\ z \end{bmatrix} = \begin{bmatrix} \bar{A} & \bar{B}_1 \\ \bar{C}_1 & \bar{D}_{11} \end{bmatrix} \begin{bmatrix} \tilde{x} \\ w \end{bmatrix} \quad (17)$$

then the following statements are equivalent:

i) *$\|T_{zw}(s)\|_\infty < \gamma$, where $T_{zw}(s) = \bar{D}_{12} + \bar{C}_1(sI - \bar{A})^{-1}\bar{B}_1$ is the transfer function of the system from w to z;*

ii) *there exists a positive definite matrix Q such that*

$$\begin{bmatrix} \bar{A}^T Q + Q\bar{A} & Q\bar{B}_1 & \bar{C}_1^T \\ \bar{B}_1^T Q & -\gamma I & \bar{D}_{11}^T \\ \bar{C}_1 & \bar{D}_{11} & -\gamma I \end{bmatrix} < 0. \quad (18)$$

PRELIMINARIES

There are two important lemmas before LMI introduction;

Lemma 3. *Suppose L_1 and L_2 are the matrices satisfying $\ker L_1 = 0$ and $\ker L_2 = 0$. Then for every matrix L_3 there exists a solution L_4 to*

$$L_1^* L_4 L_2 = L_3$$

Lemma 4. ***Elimination lemma*** *Let matrices $L_1 \in R^{nxm}$, $L_2 \in R^{kxn}$, and $L_3 = L_3^T \in R^{nxn}$ be given matrices. Consider the set of matrices $\varphi(L_1, L_2, L_3) = \{L_4 \in R^{mxk} : L_1 L_4 L_2 + (L_1 L_4 L_2)^T + L_3 < 0\}$.*

Then the following statements are equivalent:
(i) $\varphi(L_1, L_2, L_3) \neq \phi$.
(ii) The following conditions hold:
$L_1^{\perp T} L_3 L_1^{\perp} < 0$ or $L_1 L_1^T > 0$,
$L_2^{\perp T} L_3 L_2^{\perp} < 0$ or $L_2^T L_2 > 0$.

LMIs FOR H_∞ CONTROL

Using the elimination Lemma and following an algebraic procedure the following necessary and sufficient conditions for the H_∞ control problem can be obtained: There exists a controller that solves the fixed order H_∞ control problem if and only if there exist positive definite matrices X and Y such that

$$\begin{bmatrix} \begin{bmatrix} B_2 \\ D_{12} \\ 0 \end{bmatrix}^{\perp T} & \begin{matrix} 0 \\ I \end{matrix} \end{bmatrix} \begin{bmatrix} AX + XA^T & XC_1^T & B_1 \\ C_1^T X & -\gamma I & D_{11} \\ B_1^T & D_{11}^T & -\gamma I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} B_2 \\ D_{12} \\ 0 \end{bmatrix}^{\perp T} & \begin{matrix} 0 \\ I \end{matrix} \end{bmatrix}^T < 0 \quad (19)$$

$$\begin{bmatrix} \begin{bmatrix} C_2^T \\ D_{21}^T \\ 0 \end{bmatrix}^{\perp T} & \begin{matrix} 0 \\ I \end{matrix} \end{bmatrix} \begin{bmatrix} A^T Y + YA & YB_1 & C_1^T \\ B_1^T Y & -\gamma I & D_{11}^T \\ C_1 & D_{11} & -\gamma I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} C_2^T \\ D_{21}^T \\ 0 \end{bmatrix}^{\perp T} & \begin{matrix} 0 \\ I \end{matrix} \end{bmatrix}^T < 0 \quad (20)$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0 \quad (21)$$

$$rank \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \leq n_p + n_c \quad (22)$$

The rank constraint exists whenever the order of the controller is smaller than the order of the plant. The relation $Rank(I - XY) \leq n_c$ can be written as

$$Rank \begin{bmatrix} X & I \\ I & Y \end{bmatrix}$$

$$= Rank \begin{bmatrix} I & 0 \\ -X^{-1} & I \end{bmatrix} \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \begin{bmatrix} I & X^{-1} \\ 0 & I \end{bmatrix}.$$
$$= Rank \begin{bmatrix} X & 0 \\ 0 & Y - X^{-1} \end{bmatrix}.$$
$$\leq Rank(Y - X^{-1}) + Rank(X)$$

and it can be obtained that (Griogriadis and Skelton, 1996)

$Rank(X) = n_p$, $Rank(Y - X^{-1}) = Rank(Y_{12}Y_{22}^{-1}Y_{12}^T) \le n_c$.

Then by introducing the notation

$Q = \begin{bmatrix} Y & Y_{12} \\ Y_{12}^T & Y_{22} \end{bmatrix}$ and $Q^{-1} = \begin{bmatrix} X & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix}$,

where $X, Y \in R^{n_p x n_p}$ and $X_{22}, Y_{22} \in R^{n_c x n_c}$ and inserting the expressions for the closed-loop matrices in the bounded real lemma condition, the following BMI formulation of the H_∞ control problem can be obtained: Find a parameter matrix $Q > 0$ and a controller matrix $\hat{K}$ such that

$$\begin{bmatrix} (\tilde{A} + \tilde{B}_2\hat{K}\tilde{C}_2)^TQ + Q(\tilde{A} + \tilde{B}_2\hat{K}\tilde{C}_2) & Q(\tilde{B}_1 + \tilde{B}_2\hat{K}\tilde{D}_{21}) & (\tilde{C}_1 + \tilde{D}_{12}\hat{K}\tilde{C}_2)^T \\ (\tilde{B}_1 + \tilde{B}_2\hat{K}\tilde{D}_{21})^TQ & -\gamma I & (\tilde{D}_{11} + \tilde{D}_{12}\hat{K}\tilde{D}_{21})^T \\ (\tilde{C}_1 + \tilde{D}_{12}\hat{K}\tilde{C}_2) & (\tilde{D}_{11} + \tilde{D}_{12}\hat{K}\tilde{D}_{21}) & -\gamma I \end{bmatrix} < 0$$

DIRECTIONAL ALTERNATING PROJECTION METHOD WITH SP DESIGN

The aim is to minimize $\mathrm{Tr}(T + S)$ subject to

$\begin{bmatrix} T & (X - X_0) \\ (X - X_0) & I \end{bmatrix} \ge 0$,

$\begin{bmatrix} S & (Y - Y_0) \\ (Y - Y_0) & I \end{bmatrix} \ge 0$

where

$(X, Y) \in \Gamma_{convex}$,

$T, S \in \mathtt{S}^n$

We denote the minimizing solutions by (X^*, Y^*); that is, the projection onto Γ_{convex} is written as (Scherer, et. al. 1997)

$(X^*, Y^*) = P_{\Gamma_{convex}}(X_0, Y_0)$.

In addition to the above LMI constraint sets, we seek to compute the orthogonal projection onto the nonconvex constraint set Z_{n_c}. To this end, define the following sets in the space of symmetric matrices

$\mathtt{D} = \{Z \in \mathtt{S}^{2n} : Z = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}, X, Y \in \mathtt{S}^n\}$,

$\mathtt{P} = \{Z \in \mathtt{S}^{2n} : Z \ge -J\}$,

$\mathtt{R}_k = \{Z \in \mathtt{S}^{2n} : rank(Z + J) \le k\}$,

where $k = n_p + n_c$ and

$J = \begin{bmatrix} 0 & I_{n_p} \\ I_{n_p} & 0 \end{bmatrix} \in \mathtt{S}^{2n}$.

Then the connection between Z_{n_c} and $\mathtt{D}, \mathtt{P}$ and $\mathtt{R}_k$ is

$(X, Y) \in \mathrm{Z}_{n_c} \Leftrightarrow \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \in \mathtt{D} \cap \mathtt{P} \cap \mathtt{R}_{n_p + n_c}$.

Note that the sets $\mathtt{D}$ and $\mathtt{P}$ are closed convex sets, where $\mathtt{R}_k$ is the only nonconvex set.

Theorem 1. *Let* $Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^T & Z_{22} \end{bmatrix} \in \mathtt{S}^{2n}$

The orthogonal projection, $Z^* = P_{\mathtt{D}}Z$ *of* Z *onto the set* $\mathtt{D}$ *is given by (Scherer, et. al. 1997)*

$Z^* = \begin{bmatrix} Z_{11} & 0 \\ 0 & Z_{22} \end{bmatrix} \in \mathtt{S}^{2n}$

The orthogonal projection onto the set $\mathtt{P}$ followed by $\mathtt{R}_k$ is provided by the following result:

Theorem 2. *Let* $Z \in \mathtt{S}^{2n}$ *and let* $Z + J = L\Lambda L^T$ *be the eigenvalue-eigenvector decomposition of* $Z + J$*, where* Λ *is the diagonal matrix of the eigenvalues and* L *is the orthogonal matrix of the normalized eigenvectors. The orthogonal projection,* $Z^* = P_{\mathtt{PR}_k}Z$ *onto the set* $\mathtt{P}$ *followed by* $\mathtt{R}_k$ *is given by*

$$Z^* = L\Lambda_k L^T - J$$

where Λ_k *is the diagonal matrix obtained by replacing the smallest* $n_p - n_c$ *eigenvalues in* $Z + J$ *by zero. (Scherer, et. al. 1997)*

If we denote this sequence of projections by $P_{\mathtt{PR}_k}Z$, then the directional alternating projection onto the set Z_{n_c} via the following sequence of iterations:

$Z_i^a = P_{\mathtt{PR}_k}Z_{i-1}$,

$Z_i^b = P_{\mathtt{D}}Z_a$,

$Z_i^c = P_{\mathtt{PR}_k}Z_b$,

$Z_i = Z_i^a + \lambda_i(Z_i^c - Z_i^a)$,

$\lambda_i = \|Z_i^a - Z_i^b\|_F^2 / Tr(Z_i^a - Z_i^c)^T(Z_i^a - Z_i^b)$

We call this step of alternating projection algorithm an inner iteration. Hence the above iteration provides the projection $P_{\mathrm{Z}_{n_c}}(X, Y)$ of (X, Y) onto Z_{n_c} .

The alternating projection algorithm for fixed order control problem can now be programmed utilizing SP for projection onto Γ_{convex} and the above inner iteration scheme for the projection onto Z_{n_c}.

The proposed procedure is the following: first find a solution that corresponds to a full controller. This is simply done by solving an LMI feasibility for the constraint set Γ_{convex}.

Next, obtain a solution that corresponds to a controller at most $n_c - 1$.This can be done via the SP problem

minimize $Tr(X + Y)$ subject to $(X, Y) \in \Gamma_{convex}$.

The obtained solution will be the starting point for our alternating projection algorithm.

Step 1. Solve the SP problem that corresponds to a controller of order at most $n_c = n - 1$. The solution (X_0, Y_0) will be our starting point

Step 2. Consider the problem where the controller order is reduced one; i.e. set $n_c = n_c - 1$.. Compute the following iterative sequence of projections:

$(X_i^a, Y_i^a) = P_{\mathrm{Z}_{n_c}}(X_{i-1}, Y_{i-1})$

$(X_i^b, Y_i^b) = P_{\Gamma_{convex}}(X_i^a, Y_i^a)$

$(X_i^c, Y_i^c) = P_{\mathrm{Z}_{n_c}}(X_i^b, Y_i^b)$

$X_i = X_i^a + \lambda_i^X(X_i^c - X_i^a)$,

$\lambda_i^X = \|X_i^a - X_i^b\|_F^2 / Tr(X_i^a - X_i^c)^T(X_i^a - X_i^b)$

$Y_i = Y_i^a + \lambda_i^Y(Y_i^c - Y_i^a)$

$\lambda_i^Y = \|Y_i^a - Y_i^b\|_F^2 / Tr(Y_i^a - Y_i^c)^T(Y_i^a - Y_i^b)$

NUMERICAL EXAMPLES

In this section a case study on controller synthesis is given. As a model, the 2nd degree of freedom system the dynamic equations of motion

$$M_{sys}\ddot{x} + D_{sys}\dot{x} + K_{sys} = \begin{bmatrix} F \\ 0 \end{bmatrix}$$

where

$$M_{sys} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, D_{sys} = \begin{bmatrix} 0.02 & -0.01 \\ -0.01 & 0.01 \end{bmatrix}$$

$$K_{sys} = \begin{bmatrix} 8 & -4 \\ -4 & 4 \end{bmatrix}$$

is selected. The external harmonic excitation force with unit amplitude acts on the first degree of freedom. The vibration amplitude of the second degree of freedom at $\omega = 1.24$ is 40.4536 which should be minimized. The controller acts on the second degree of freedom of the system. These equations are converted to the state-variable and output equations by attaining new state variables to system variables (e.g. x_1 for x, x_2 for $\dot{x}_1, \ldots etc.$) The corresponding state space matrices are;

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -8 & 4 & -0.02 & 0.01 \\ 4 & -4 & 0.01 & -0.01 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, C_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$$

The state space assumptions for the system are:

A.1 $D_{11} = I, D_{22} = 0$.

A.2 (A, B_1) is stabilizable and (C_1, A) is detectable.

A.3 (A, B_2) is stabilizable and (C_2, A) is detectable for existence of a stabilizing K.

A.4 For ensurance of proper and realizable controller : $rank D_{12} = n_u$, $rank D_{21} = n_y$.

A.5 $D_{12}^T \begin{bmatrix} C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$. It means that $C_1 x$ and $D_{12} u$ are orthogonal so that the penalty on $z = C_1 x + D_{12} u$ includes a nonsingular penalty on the control u.

A.6 $\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}^T = \begin{bmatrix} 0 \\ I \end{bmatrix}$. It is dual to A.5 and concerns how the exogenous signal w enters P: w includes both plant disturbance and sensor noise, these are orthogonal, and the sensor noise weighting is nonsingular.

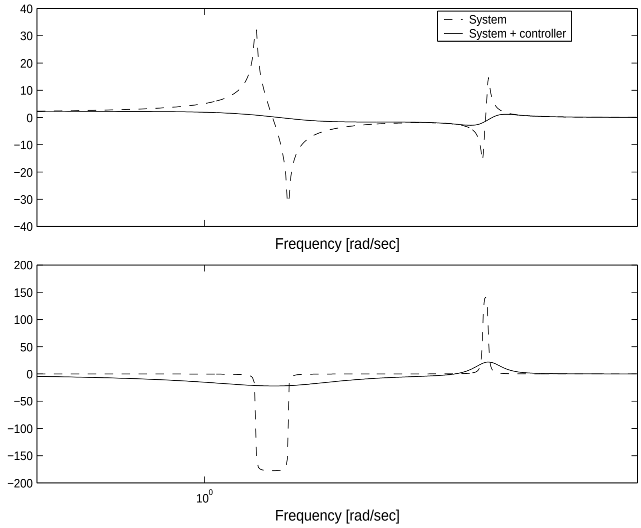

Figure 2. Full order controller.

A.7 $rank \begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix} = n_p + n_u$ and $rank \begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix} = n_p + n_y \ \forall \omega \in R$. to ensure that the optimal controller does not try to cancel poles or zeros on the imaginary axis which would result in closed-loop instability.

A.8 The system is assumed to be collocated;

$$C_2 = B_2^T$$

FULL ORDER CONTROLLER SYNTHESIS FOR THE 2-DOF SYSTEM

A 4th order controller is to be synthesized. After the optimization process; the results are: $\|T_{zw}(s)\|_\infty < \gamma_{min} = 1.9762$. The minimized vibration amplitude of the second mass at $\omega = 1.24$ is 1.1023. The frequency response with and without controller is given in Figure 2. The synthesized controller is :

$$K = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right] = \left[\begin{array}{cccc|c} -0.5947 & -1.6427 & -0.9168 & 9.6936 & -0.7911 \\ 0.5199 & -0.1866 & -1.1390 & -0.5157 & -0.1814 \\ 0.5471 & 0.7265 & -0.9587 & 3.7773 & -0.6578 \\ -0.8352 & 0.1647 & -0.6327 & -0.6283 & 0.4658 \\ \hline -0.4317 & -0.2748 & 0.3179 & -0.4617 & -0.9989 \end{array}\right]$$

FULL ORDER CONTROLLER SYNTHESIS FOR THE STRICT POSITIVE REALNESS OF THE 2-DOF SYSTEM

A 4th order controller which makes the closed loop system strict positive real is to be synthesized. After

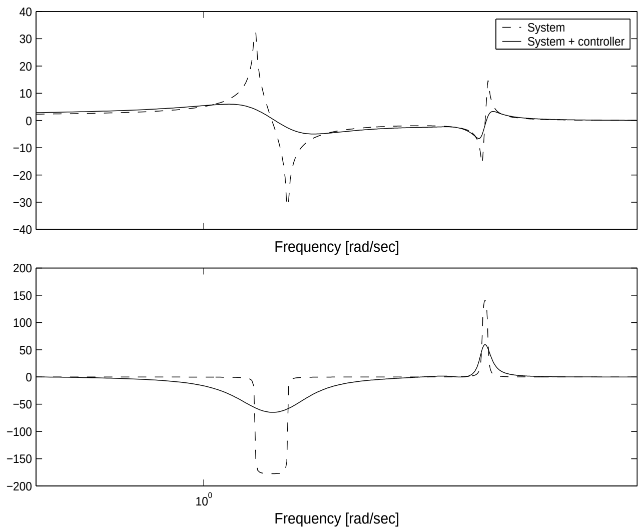

Figure 3. Full order controller with strict positive real closed loop system

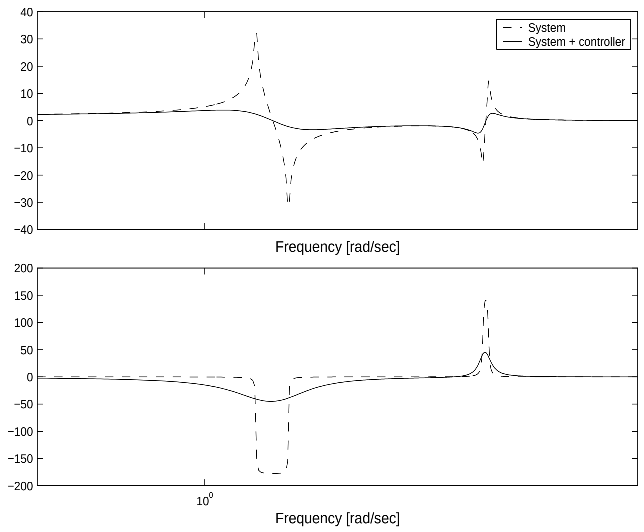

Figure 4. Reduced order controller.

the optimization process; the results are: $\|T_{zw}(s)\|_\infty < \gamma_{min} = 16.5$. The minimized vibration amplitude of the second mass at $\omega = 1.24$ is 1.5854. The frequency response with and without controller is given in Figure 3. The synthesized controller is :

$$K = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right] = \left[\begin{array}{cccc|c} -1.6328 & -2.7734 & 18.2239 & -7.4763 & 16.7006 \\ 0.3790 & -0.3237 & 0.1865 & -2.0666 & 1.4899 \\ 1.7444 & 2.2142 & -21.3240 & 21.5843 & -19.3678 \\ -1.4635 & -0.4372 & 7.1980 & -7.9203 & 7.649 \\ \hline -0.7265 & -0.6763 & 9.4424 & -8.8429 & 8.0904 \end{array}\right]$$

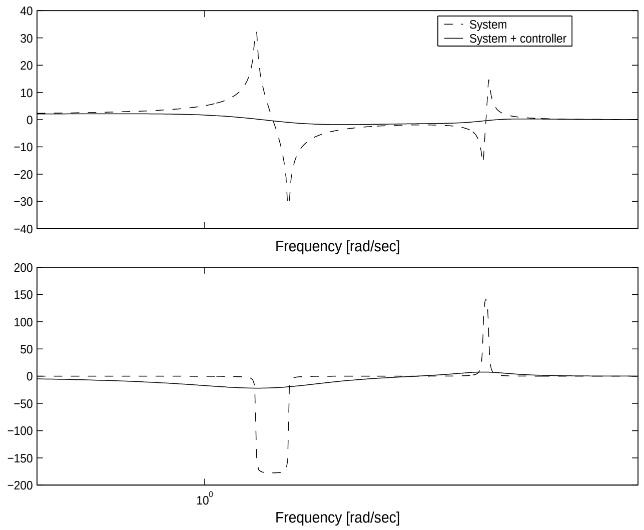

Figure 5. Decentralized controller.

REDUCED ORDER CONTROLLER SYNTHESIS FOR THE 2-DOF SYSTEM

A 2nd order controller is to be synthesized. After the optimization process; the results are: $\|T_{zw}(s)\|_\infty < \gamma_{min} = 2.0962$. The minimized vibration amplitude of the second mass at $\omega = 1.24$ is 1.3002. The frequency response with and without controller is given in Figure 4. The synthesized controller is : $K = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right] =$

$$\left[\begin{array}{cc|c} -14.1607 & -8.7575 & 0.1726 \\ -8.1012 & -9.5012 & 0.1178 \\ \hline 3.9614 & 5.6944 & -0.6410 \end{array}\right]$$

DECENTRALIZED CONTROLLER SYNTHESIS FOR A 2-DOF SYSTEM

A 4th order decentralized controller is to be synthesized. After the optimization process; the results are: $\|T_{zw}(s)\|_\infty < \gamma_{min} = 2.8212$. The minimized vibration amplitude of the second mass at $\omega = 1.24$ is 1.0291. The frequency response with and without controller is given in Figure 5. The synthesized controller is:

$$K = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right] = \left[\begin{array}{cccc|cc} -0.5776 & 1.0747 & 0 & 0 & -0.1540 & 0 \\ -1.2692 & -0.7737 & 0 & 0 & -0.1555 & 0 \\ 0 & 0 & -1.9347 & 6.8161 & 0 & -0.0872 \\ 0 & 0 & -1.6160 & -3.4287 & 0 & -0.4106 \\ \hline 0.2448 & 0.1918 & 0 & 0 & -0.7745 & 0 \\ 0 & 0 & -0.0354 & -1.1802 & 0 & -1.1291 \end{array}\right]$$

CONCLUSION

In this paper, solution methods for the H_∞ control problem are presented using linear matrix inequalities (LMI's). Synthesis of full, decentralized or reduced order controllers is realized for this purpose. Moreover, strict positive realness of the closed loop system combined with the H_∞ control problem is also considered in the controller synthesis problem. The constraints on the system and controller transfer functions increase the H_∞ norm and give less effective results for the system performance. Positive realness and robustness of the synthesized controllers will be the next topics of the future research.

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