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PARAMETER DEPENDENT CONTROL LAW DESIGN WITH APPLICATIONS TO NONLINEAR SYSTEMS

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ABSTRACT

This paper presents a general framework for the control of nonlinear systems of the form $E_x(r)\dot{x} = f(x,u)$, $E_y(r)y = h(x,u)$ using techniques developed for linear parameter-varying systems. The stability characteristics and the control law of the system are obtained with both classical Lyapunov considerations and linear matrix inequalities (LMI). In the synthesis part, multi-convexity criteria is used to construct a linear finite dimensional controller, whose state-space entries can also depend continuously on parameters r, such that the closed-loop system is exponentially stable and achieves good performance with respect to variations in these parameters. While the performance and the stability specifications of the system are achieved, the region of attraction of the system is considered as a design criterion. Both the physical constraints and the chosen control law act on the determination of the region of attraction of the system. It is shown that the region of attraction for a given control law can be systematically derived by LMI-based approaches. All developed methods are then applied on a cart and inverted pendulum system. Generalizations from the results are concluded for both system with LPV control and the system with the classical linearization method.

INTRODUCTION

Designing a controller for systems with widely varying nonlinear dynamics is a major area of research in control theory. Gain-scheduling is a technique widely used to control such systems in a variety of engineering applications. However, in the design of the gain-scheduled controller, the nonlinear system is divided into some operating points at which a linear time invariant (LTI) controller is found and a global controller is then computed by interpolation between these points. As it is seen, interpolation step constitutes a drawback for this controller method. Therefore, a new control method which does not include such disadvantages is needed. At the beginning of 1990's, linear parameter varying (LPV) control method has been introduced by Shamma and Athans (Shamma and Athans, 1991) to overcome all these difficulties. This method includes linear time-varying plant models whose state-space description is a fixed function of some parameter vector r. The parameter vector $r(t) \in \Re^n$ is not uncertain and assumed that the value of $r(t)$ is known in real-time which give real-time information on variations in the plant characteristics, that is, the model at time t is assumed to be obtained only at that instant t through the parameter r, not beforehand.

In 1991, Shamma and Athans has studied the LPV systems by considering them as LTI systems in which slowly varying parameters are used. Packard, in 1991 and 1992, "self scheduled" the controller and tried to obtain the stability and high performance along all trajectories $r(t)$. The study in (Gahinet et. al., 1994) has proposed an LMI-based test for the robust stability/performance of linear systems with uncertain real parameters. Beside LMI-based techniques, the contribution of the convexity approach to LPV problems has been firstly seen in the papers of Iwasaki and Skelton (Iwasaki and Skelton, 1994). In this paper, the covariance control approach is introduced at first. Other approaches of convexity involving LMIs and Riccati equations include parametric Lyapunov functions (Yu and Sideris, 1997), LFT and

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polytopic design (Kajiwara et. al., 1999), and LQR solution techniques for a cart-pendulum system with the consideration of maximized stability region (Seto and Sha, 1999).

All of these studies have shown that the LPV system and nonlinear system studies have a somewhat very close relationship since the theoretical developments in LPV systems occurred at the same rate as the developments in the nonlinear systems. The available LPV synthesis techniques allow the construction of the global control law as a whole entity for all admissible r. They furthermore provide theoretical guarantees in terms of both stability and performance in the presence of fast-time evolutions of the parameters. But an important issue in the design of control systems involves the question to what extent the stability and performance of the controlled system is robust against perturbations and uncertainties in the parameters of the system. The main analysis approach for this assessment of stability and performance makes use of the method of Lyapunov which is summarized as follows; an equilibrium point is stable if all solutions starting at close distances from that point stay nearby; otherwise, it is unstable. It can be shown that the differential equation

$$\dot{x} = Ax(t) \tag{1}$$

is stable if and only if, there exists a positive definite matrix $P = P^T$ and the condition $A^T P + PA < 0$ is satisfied, which we now call as a Lyapunov inequality on P (Khalil, 1996).

The objective in this paper is to develop a design methodology that ensures the stabilization of the nonlinear systems with constraints while maximizing the "*volume*" of the region in the state-space $\Re^n$ over which the ellipsoidal method is applied. The approach heavily depends on the LMI technique. Throughout the paper, the system that will be analyzed with respect to the stability is given by:

$$E_x(r)\dot{x} = A(r)x + B(r)u \tag{2}$$
$$E_y(r)y = C(r)x + D(r)u. \tag{3}$$

The first step in controller design is to obtain a linear description or approximation for the nonlinear plant that involves time-varying parameters. Time-varying parameters can be defined as,

$$r(t) = \left[r_1(t)\ r_2(t) \cdots\ r_p(t) \right]^T \tag{4}$$

whose time variations are constrained by

$$r(t) \in S \subset \Re^p \tag{5}$$
$$\dot{r}(t) \in T \subset \Re^p. \tag{6}$$

LINEAR PARAMETER VARYING SYSTEMS

In state-space form, an LPV system is described by

$$\begin{aligned} \dot{x} &= A(r)x + B(r)u \\ y &= C(r)x + D(r)u \end{aligned} \tag{7}$$

where $x \in \Re^n$ is the plant state, $u \in \Re^{m_2}$ is the control input, $y \in \Re^{p_2}$ is the measured output and $r = r(t)$ denotes a time-varying parameter vector. The parameter vector is called as the admissible parameter trajectories which are continuously differentiable time-varying vectors having the form of

$$\begin{aligned} P \triangleq \{ r : \Re^+ \to \Re^q : r(t) \in S \\ \text{and } \dot{r}(t) \in T\ , \forall t \geq 0 \} \end{aligned} \tag{8}$$

where

$$S \triangleq \{ r \in \Re^q : \underline{r}_\alpha \leq r_\alpha \leq \bar{r}_\alpha, \forall \alpha = 1 : q \} \tag{9}$$

and

$$T \triangleq \left\{ d \in \Re^q : \underline{d}_\alpha \leq \dot{r}_\alpha \leq \bar{d}_\alpha, \forall \alpha = 1 : q \right\} \tag{10}$$

The vertices of S and T can be denoted as:

$$\begin{aligned} &S_{vex} \triangleq \{ r : r_\alpha = \underline{r}_\alpha \text{ or } \bar{r}_\alpha, \forall \alpha = 1 : q \} \\ &\text{and} \\ &T_{vex} \triangleq \{ \dot{r}_\alpha = \underline{\dot{r}}_\alpha \text{ or } \bar{\dot{r}}_\alpha, \forall \alpha = 1 : q \} \end{aligned} \tag{11}$$

However, as indicated before, throughout the study, the system of interest is

$$\begin{aligned} E_x(r)\dot{x} = A(r)x + B(r)u \\ E_y(r)y = C(r)x + D(r)u \end{aligned} \tag{12}$$

where E's are invertible and all matrices are affine in r_i's, i.e.,

$$\begin{aligned} &\Gamma(r) = \Gamma_0 + \textstyle\sum_{i=1}^n r_i \Gamma_i\ , \text{ where} \\ &\Gamma :\ E, A, B, C, D. \end{aligned} \tag{13}$$

Thus, at this point, the difference between the general form of LPV systems, (7), and (12) must be stated. Since, when the system (12) is premultiplied by $E_i^{-1}(r)$, $i = x, y, z$, the obtained system is exactly an LPV system of the form (7) and therefore one may suppose that the classical solution techniques must be applied for this special case. However, since it was assumed, at the beginning, all the techniques applied are valid for convex systems and the premultiplication terms, E_i's, corrupt affinity and so convexity, the results used for general form of LPV systems cannot be used for the systems of the form (12). Therefore, throughout the study, the LPV systems of the form (12) will be analyzed and the generalized solutions will be obtained.

The following section deals with the analysis of the system with respect to the stability characteristics.

ANALYSIS

The analysis part contains the case where there is no control input act upon the system (12). If $u(t) = 0$ is replaced in (12), we end up with

$$E(r)\dot{x} = A(r)x. \tag{14}$$

The following theorem will give us sufficient conditions for the above system to be asymptotically stable.

Theorem 1. *The system (14) is asymptotically stable if there exists $X(r) = X(r)^T$ such that,*

$$X(r) > 0\ ,\ \forall r \in S \tag{15}$$

and

$$\begin{gathered} E(r)X(r)A(r)^T + A(r)X(r)E(r)^T \\ -E\dot{X}E^T(r) < 0\ ,\ \forall r \in S,\ ,\ \forall \dot{r} \in T \end{gathered} \tag{16}$$

However, this theorem states that there are infinitely many solutions satisfying LMI's (15) and (16) under the domain of (S, T). Since we are dealing with convex functions, if the convexity of the system are guaranteed by applying multi-convexity criteria (Gahinet et. al., 1994), then it will be sufficient, instead of looking at the all points of the set, to check for the corner points of the set $S \times T$ for the system to be finite dimensional. The multi-convexity criteria states that a system is convex if the second derivative of the system with respect to the system parameter is greater than or equal to zero. Therefore, for the system (14) to be asymptotically stable, the following LMI's must be satisfied,

$$X(r) > 0\ , \forall r \in S_{vex}, \tag{17}$$

$$\begin{gathered} E(r)X(r)A(r)^T + A(r)X(r)E^T(r) \\ -E(r)\dot{X}(r)E(r)^T < 0, \\ \forall (r, \dot{r}) \in S_{vex} \times T_{vex}, \end{gathered} \tag{18}$$

$$\begin{gathered} E_\alpha X_\alpha A^T + E_\alpha X A_\alpha^T + E X_\alpha A_\alpha^T + \\ A_\alpha X_\alpha E^T + A_\alpha X E_\alpha^T + A X_\alpha E_\alpha^T - \\ E_\alpha \dot{X} E_\alpha^T \geq 0\ ,\ \forall \alpha = 1, \ldots, n. \end{gathered} \tag{19}$$

Since the LMI conditions that make the system (14) asymptotically stable are obtained, we can proceed to the next section which mainly deals with obtaining a controller satisfying asymptotic stability.

SYNTHESIS

Our goal is to design a controller that satisfies asymptotic stability with L_2-gain γ (Wu et. al., 1996). While considering controller synthesis for the system (12), we assume all r's and $\dot{r}$'s are available for feedback. If the system's state matrix is considered as $\tilde{x} = \begin{bmatrix} x_1\ x_2\ \ldots\ x_n \end{bmatrix}$, the system can be controlled by fully- or partly-feeding back the state matrix. We will analyze the system with dynamic output feedback control.

In dynamic output feedback control, the goal is to design a controller of the form

$$\begin{aligned} \dot{x}_c &= A_c(r, \dot{r})x_c + B_c(r, \dot{r})y, \\ u &= C_c(r, \dot{r})x_c + D_c(r, \dot{r})y. \end{aligned} \tag{20}$$

such that the closed loop system (12), as seen in Figure 1, is asymptotically stable with L_2-gain γ.

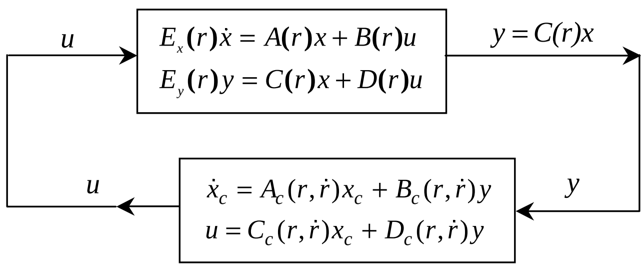

Figure 1. Block diagram for dynamic output feedback control.

For simplification $D(r)$ is assumed to be identically zero and the following theorem is applied.

Theorem 2. *There exists a controller of the form (20) that asymptotically stabilizes system (12) if there exists $X = X^T : P \rightarrow \Re^{n \times n}$, $Y = Y^T : P \rightarrow \Re^{n \times n}$, $F : P \rightarrow \Re^{m_2 \times n}$ and $G : P \rightarrow \Re^{n \times p_2}$ such that (Köse, 2001),*

$$\begin{gathered} AXE_x^T + E_xXA^T + B_2FE_x^T + \\ E_xF^TB_2^T - E_x\dot{X}E_x^T < 0 \end{gathered} \tag{21}$$

$$\begin{gathered} A^TYE_x + E_x^TYA + GC_2 + C_2^TG^T + \\ \dot{E}_x^TYE_x + E_x^T\dot{Y}E_x + E_x^TY\dot{E}_x < 0 \end{gathered} \tag{22}$$

where $\forall(r,\dot{r}) \in S \times T$ *and*

$$\begin{bmatrix} X(r) & I \\ I & E(r)^T Y(r) E(r) \end{bmatrix} > 0 \;,\; \forall r \in P \tag{23}$$

In this case, the controller in (20) can be given by the following definitions:

$$\begin{aligned} D_c &= 0 \\ C_c(r) &= F X^{-1} \\ B_c(r) &= -Z^{-1} G \\ A_c(r,\dot{r}) &= Z^{-1} E_x^T Y [A + B_2 C_c] - B_c E_y^{-1} C_2 + \\ & \quad Z^{-1} A^T E_x^{-T} X^{-1} + Z^{-1} X^{-1} \dot{X} X^{-1} \end{aligned} \tag{24}$$

where $Z(r) = E_x(r)^T Y(r) E_x(r) - X(r)^{-1} > 0$.

Again for the finite dimensional form, we must apply the multi-convexity criteria for the conditions to be satisfied at $(S \times T)_{vex}$. The multi-convexity condition of (21) is the same as (19). For (22), enforce

$$\begin{aligned} & A_\alpha^T (2Y_\alpha E + Y E_\alpha) + (2Y_\alpha E + Y E_\alpha)^T A_\alpha + \\ & (A_\alpha^T Y + 2A^T Y_\alpha) E_\alpha + E_\alpha^T (A_\alpha^T Y + 2A^T Y_\alpha)^T + \\ & \dot{E}^T Y_\alpha E_\alpha + E_\alpha^T \dot{Y} E_\alpha + E_\alpha^T Y_\alpha \dot{E} \geq 0 \;, \\ & \forall(r,\dot{r}) \in S_{vex} \times T_{vex} \;,\; \forall \alpha = 1:q \end{aligned} \tag{25}$$

and the second derivative of (23) results in

$$\begin{aligned} & E_\alpha^T Y_\alpha E + E_\alpha^T Y E_\alpha + E^T Y_\alpha E_\alpha \leq 0 \;, \\ & \forall r \in S_{vex} \;,\; \forall \alpha = 1:q. \end{aligned} \tag{26}$$

Up to this point, we did not put any limitations to the system's stability characteristics. However, in real life, both the operating conditions and performance conditions, such as physical limitations (length or angle range) and power limit of a motor, constraint a system. Therefore, the following section analyzes the limitations acting on the system and aims to construct a region of attraction under all these limitations.

THE REGION OF ATTRACTION WITH LINEAR CONSTRAINTS

The invariant region or the region of attraction of a linear control system will be restricted by the constraints imposed to the system. The system can only evolve in the feasible region in the state space, where no constraints will be violated. Thus an invariant region has to be a subset of the feasible region. An invariant region is defined as;

Definition 1. *If we denote the state trajectory originating at* x_0 *at time* $t = 0$ *by* $\psi(t, x_0)$, *then for a set* $S \subset \Re^n$, *we say* S *is invariant with respect to a system if* $\forall x_0 \in S$ *implies* $\psi(t, x_0) \in S$ *for all* $t \geq 0$.

We then have the following invariance property:

Theorem 3. *If a matrix defined as* $X = X^T > 0$ *is the Lyapunov matrix of a system that makes it asymptotically stable, then the ellipsoid*

$$\Psi_{X^{-1}} \triangleq \{x \in \Re^n : x^T X^{-1} x \leq 1\} \tag{27}$$

is invariant with respect to the defined system.

If the physical plant that we are interested in is defined as $\dot{x} = (x, u(x,t))$, then the state constraints acting on the system are $q_1(x) \leq 0, ..., q_l(x) \leq 0$ and the control constraints are $p_1(u) \leq 0, \ldots, p_r(u) \leq 0$. The state and control constraints together give the physical constraints to the physical system. The safety of the system is concerned with the operation of the physical system without violating these physical constraints.

However, we are interested with LMI's and therefore all these physical limitations must be introduced to the system as LMI's. For example, a constraint defined as $|x_i| \leq \bar{x}_i$ is put into an LMI form by the following manipulations

$$|x_i| \leq \bar{x}_i \Longrightarrow |e_i^T x_i| \leq \bar{x}_i \Longrightarrow x^T e_i e_i^T x \leq \bar{x}_i^2 \tag{28}$$

$$\Longrightarrow x^T \frac{e_i e_i^T}{\bar{x}_i^2} x \leq 1 \tag{29}$$

and since our invariant region must be within this limitation, using the invariance property (Öncel, 2001)

$$\frac{e_i e_i^T}{\bar{x}_i^2} \;\leq\; X^{-1} \Longrightarrow X \frac{e_i e_i^T}{\bar{x}_i^2} X \leq X \tag{30}$$

$$\Longleftrightarrow -X + X e_i (\bar{x}_i^2)^{-1} e_i^T X \leq 0 \tag{31}$$

$$\Longleftrightarrow \begin{bmatrix} -X & X e_i \\ e_i^T X & -\bar{x}_i^2 \end{bmatrix} \leq 0 \tag{32}$$

In these manipulations, e_i is a column matrix denoting which variable of the state matrix to be constrained and the last inequality is a result of Schur complement.

To determine the invariant set under dynamic output feedback control, we are aware of the fact that there is a relationship between X and Y parameters as $\widetilde{Y} - X^{-1} \geq 0$ where, as before, $\widetilde{Y} = E^T Y E$. In that case, we have a Lyapunov matrix P with the following property

$$P = \begin{bmatrix} X & X \\ X & X + Z^{-1} \end{bmatrix}^{-1} = \begin{bmatrix} Y & * \\ * & * \end{bmatrix} > 0 \tag{33}$$

where $*$ denotes the unnecessary information. With that definition of P, our invariant stability region takes the following form (see Figure 2);

$$\Psi_P = \left\{ \begin{bmatrix} x \\ x_c \end{bmatrix}^T \begin{bmatrix} X & X \\ X & X+Z^{-1} \end{bmatrix}^{-1} \begin{bmatrix} y \\ y_c \end{bmatrix} \leq 1 \right\}. \tag{34}$$

Then the sets $\Psi_{X(r)}$, $\Psi_{\widetilde{Y}(r)}$, and $\Psi_{P(r)}$ are invariant in the sense that given any $\widehat{x}(0) \in \Psi_{P(r)}$, *the ensuing state trajectory* satisfies $x(t) \in \Psi_P$ for all $t \geq 0$. It is also clear that all initial conditions, $\widehat{x}(0) = \begin{bmatrix} x_0^T & 0 \end{bmatrix}^T$, lie in $\Psi_{P(r)}$ if and only if

$$\begin{aligned} & x(0) \in \Psi_{\widetilde{Y}(r)} \triangleq \\ & \textstyle\bigcap_{r \in S} \{x \in \Re^n : x^T \widetilde{Y}(r) x \leq 1\} = \\ & \textstyle\bigcap_{r \in \Re_{vex}} \{x \in \Re^n : x^T \widetilde{Y}(r) x \leq 1\}. \end{aligned} \tag{35}$$

Moreover, as discussed in (Ghaoui and Scorletti, 1996), if $\widehat{x}_0 \in \Psi_{P(r)}$, then $x(t) \in \Psi_{X(r)^{-1}}$ and $x_c(t) \in \Psi_{(X+Z^{-1})^{-1}}$ for all $t \geq 0$, where $\Psi_{X^{-1}}$ is defined in (27) and

$$\begin{aligned} & \Psi_{(X+Z^{-1})^{-1}} \triangleq \\ & \textstyle\bigcap_{r \in S} \{x \in \Re^n : x^T (X(r) + Z(r)^{-1})^{-1} x \leq 1\}. \end{aligned} \tag{36}$$

In other words, as seen in Figure 2, for all initial conditions starting in the invariant region of $\Psi_{P(r)}$, the state trajectories must not leave the sets $\Psi_{X^{-1}}$ and $\Psi_{(X+Z^{-1})^{-1}}$ for asymptotic stability.

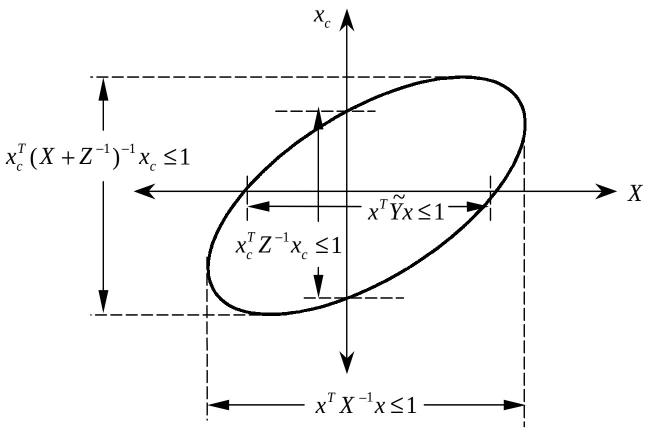

Figure 2. The invariant set and the necessary notations under dynamic output feedback control.

The state and control constraints acting on the dynamic output feedback control can be described as follows;

$r_\alpha(x_i) \in [\underline{r}_\alpha, \bar{r}_\alpha]$ **if** $|x_i| \leq \bar{x}_i$: The condition $|x_i(t)| \leq \bar{x}_i$ for all $t \geq 0$ is satisfied if $\Psi_{X^{-1}}$ lies inside $\{x \in \Re^n : |x_i| \leq \bar{x}_i\}$ which is introduced to the system, as above, by the following LMI;

$$\begin{bmatrix} -X(r) & X e_{n,i} \\ e_{n,i}^T X & -\bar{x}_i^2 \end{bmatrix} \leq 0 \;\forall r \in S_{vex}. \tag{37}$$

$\mathbf{r}_\alpha(\mathbf{u}_j) \in [\underline{r}_\alpha, \bar{r}_\alpha]$ **if** $|u_j| \leq \bar{u}_j$: Since the controller state x_c never leaves the region $\Psi_{(X+Z^{-1})^{-1}}$ and $u = C_c(r) x_c = F X^{-1} x_c$, by the same manipulations as above we need to enforce the condition

$$\begin{bmatrix} -X & F^T e_{m,j} & I \\ e_{m,j}^T F & -\bar{u}_j^2 & 0 \\ I & 0 & -Y \end{bmatrix} \leq 0. \tag{38}$$

APPLICATION TO A CART AND INVERTED PENDULUM SYSTEM

The inverted pendulum is a very popular experiment used for educational purposes in modern control theory. As shown in Figure 3, the physical system consists of a cart, driven by a AC motor, and a pendulum attached to the cart. The cart can move along a horizontal track, and the pendulum is able to rotate freely in the range of $[-60°, 60°]$ with respect to vertical in the vertical plane parallel to the track. The control objective is to bring the pendulum to the upper unstable equilibrium position by moving the cart on the horizontal axis.

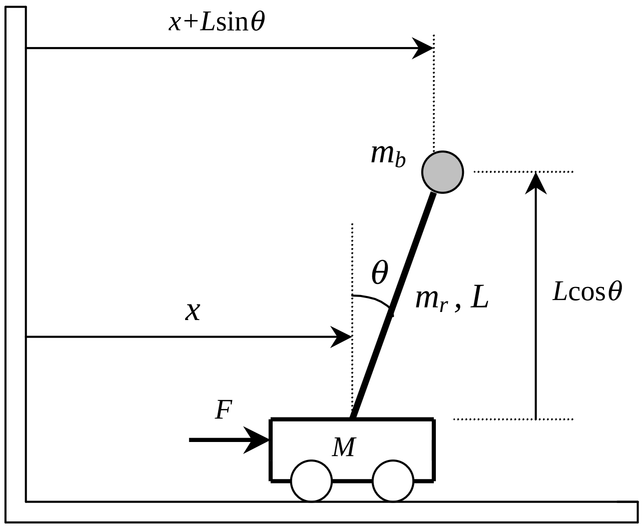

Figure 3. The cart and inverted pendulum model.

The system's governing equations are:

$$\begin{aligned} (\tfrac{m_r}{2}+m_b)L\cos\theta\ddot{x}+(\tfrac{m_r}{3}+m_b)L^2\ddot{\theta}&=(m_b+\tfrac{m_r}{2})gL\sin\theta \\ (M+m_r+m_b)\ddot{x}+(m_b+\tfrac{m_r}{2})(L\cos\theta\ddot{\theta}-L\dot{\theta}^2\sin\theta)&=F \end{aligned} \tag{39}$$

where m_b, m_r and M stands for the masses of the bob of the pendulum, rod, and the cart, respectively, L is the length of the pendulum, g stands for the gravitational acceleration, F is the force applied to the cart.

This highly-nonlinear system must be turned into an LPV form by newly-defined parameters. The parameters of the form $r_1 \triangleq \cos\theta$, $r_2 \triangleq \sin\theta/\theta$, $r_3 \triangleq \sin\theta\dot{\theta}$ and the state vector $x = \begin{bmatrix} x \; \theta \; \dot{x} \; \dot{\theta} \end{bmatrix}$ put (39) into a state space form as described in (12);

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (m_b+\frac{m_r}{2})Lr_1 & (\frac{m_r}{3}+m_b)L^2 \\ 0 & 0 & (M+m_r+m_b) & (m_b+\frac{m_r}{2})Lr_1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & (m_b+\frac{m_r}{2})gLr_2 & 0 & 0 \\ 0 & 0 & 0 & (m_b+\frac{m_r}{2})Lr_3 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} F. \tag{40}$$

Since the finite dimensional form of the system (40) is desired, the minimum and maximum values of the parameters must be defined for θ between $-\pi/3$ and $\pi/3$;

$$\begin{aligned} r_1 \in [1/2, 1] \text{ and } r_2 \in \left[3\sqrt{3}/2\pi, 1\right] \\ \text{for } \theta \in [-\pi/3, \pi/3], \\ r_3 \in \left[-\tfrac{\sqrt{3}}{2}\omega, \tfrac{\sqrt{3}}{2}\omega\right] \\ \text{for} (\theta, \dot{\theta}) \in [-\pi/3, \pi/3] \times [-\omega, \omega] \end{aligned} \tag{41}$$

and

$$\begin{aligned} \dot{r_1} = -r_3 \text{ and } \dot{r_2} \in [-0.3123\omega, 0.3123\omega] \\ \text{for } (\theta, \dot{\theta}) \in [-\pi/3, \pi/3] \times [-\omega, \omega]. \end{aligned} \tag{42}$$

Now, since the system is a physical plant, it includes some physical constraints, such as the AC motor has only limited power and the track has finite length, which are introduced as LMI conditions;

$$|x| \le h \;\Rightarrow \begin{bmatrix} -X(r) & Xe_{4,1} \\ e_{4,1}^T X & -h^2 \end{bmatrix} \le 0, \tag{43}$$

$$|\theta| \le \pi/3 \;\Rightarrow \begin{bmatrix} -X(r) & Xe_{4,2} \\ e_{4,2}^T X & -(\pi/3)^2 \end{bmatrix} \le 0, \tag{44}$$

$$|\dot{x}| \le 3 \;\Rightarrow \begin{bmatrix} -X(r) & Xe_{4,4} \\ e_{4,4}^T X & -w^2 \end{bmatrix} \le 0, \tag{45}$$

$$|u| \le 100 \;\Rightarrow \begin{bmatrix} -X(r) & F^T(r) \\ F(r) & -\overline{u}^2 \end{bmatrix} \le 0, \tag{46}$$

$\forall r \in S_{vex}$, where $e_{4,1} = \begin{bmatrix} 1 \; 0 \; 0 \; 0 \end{bmatrix}^T$, $e_{4,2} = \begin{bmatrix} 0 \; 1 \; 0 \; 0 \end{bmatrix}^T$ and $e_{4,3} = \begin{bmatrix} 0 \; 0 \; 1 \; 0 \end{bmatrix}^T$. Therefore our revised stability region S is described by

$$\begin{aligned} S = \{X \mid |x_1| \le 0.8,\ |x_2| \le \pi/3, \\ |x_3| \le 3,\ |u| \le 100\} \end{aligned} \tag{47}$$

With the above definitions of the physical constraints, the notion of a region of attraction will be introduced to characterize a subset of the system state from which the system stability can always be maintained.

For the following controller

$$\dot{x_c} = A_c(r, \dot{r})x_c + B_c(r, \dot{r})y \tag{48}$$

$$u = C_c(r, \dot{r})x_c + D_c(r, \dot{r})y \tag{49}$$

the performance of the system is obtained as in Figure 4;

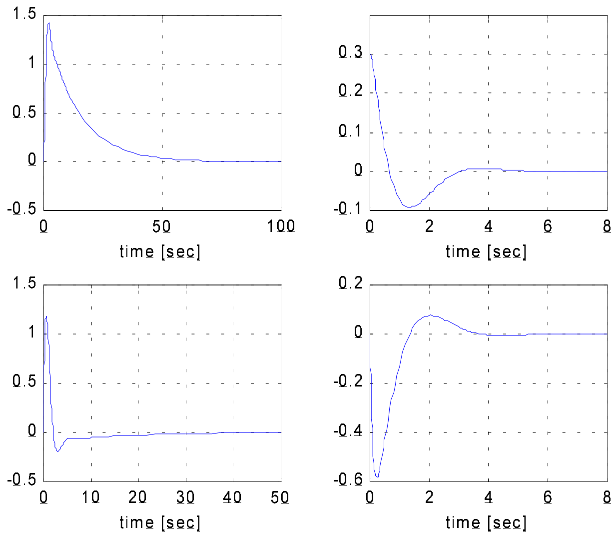

Figure 4. System performance under dynamic output feedback control for an initial angle of $\theta = 0.3$ rad.

All the performance characteristics in Figure 4 is obtained for the initial condition of $x_0 = \begin{bmatrix} 0 \; 0.3 \; 0 \; 0 \end{bmatrix}^T$. We are mostly interested with the pendulum and cart position figures since we put constraints on these parameters. As seen from the figure, the pendulum comes to the upright position after approximately 6 seconds under dynamic output feedback controller. As the left-top figure shows, the rise time of the cart position is approximately 2 sec. In the cart velocity graph, we see that the cart never exceeds the limit value, 3 m/sec. Pendulum angle (position) figure gives us the results for rise time, settling time and overshoot as $t_r = 0.8$ sec, $t_s \cong 5$ sec and $M_p = 0.0915$ rad, respectively.

Now, let's consider the input force graph. Figure 5 gives us the force graph for an initial angle of $\theta = 30^o$. In that figure, our maximum force takes the value of 15.6 N. Since the constraint imposed on the input force is 100 N, the result is acceptable.

Analyzing the stability region and the system stability with respect to whether it goes to zero or not gives us Figure 6. The figure shows us the maximized region

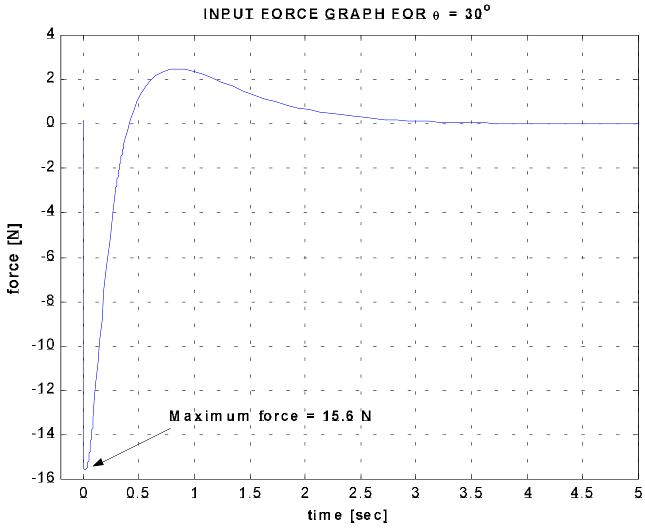

Figure 5. Input force graph for $\theta = 30^o$.

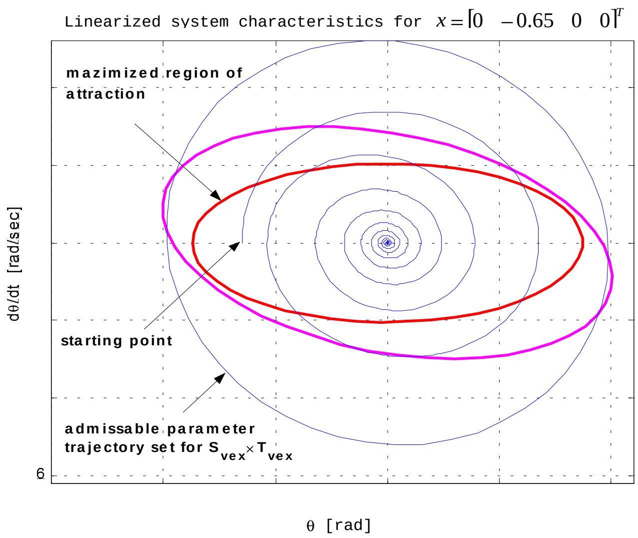

Figure 6. Invariant set and stability characteristics of the inverted pendulum system under dynamic output feedback control.

of attraction and the ellipsoids evaluated at the vertices of the admissible parameter trajectory which is defined in (8). According to our theory, for a system to be stable, a state trajectory starting in the maximized region of attraction must not leave the outermost ellipsoid of the admissible parameter trajectory. Therefore, to prove the stability of our system, we choose 3 state trajectories, one is starting at a very close point to the boundary of the maximized region of attraction. Namely, $\left[0\ 0.3\ 0\ 0\right]^T$, $\left[0\ 0.5\ 0\ 0\right]^T$ and $\left[0\ 0.8\ 0\ 0\right]^T$.The figure shows that the state trajectory reach to the origin in each cases validating our theory and thus the stability of our system under dynamic output feedback control.

Now, it is time for making comparisons between the LPV system and the linearized system. The linearized system is obtained from (39) with the classical small angle approach. The linearized system response to an initial angle of $\theta = 0.65$ rad. is seen in Figure 7.

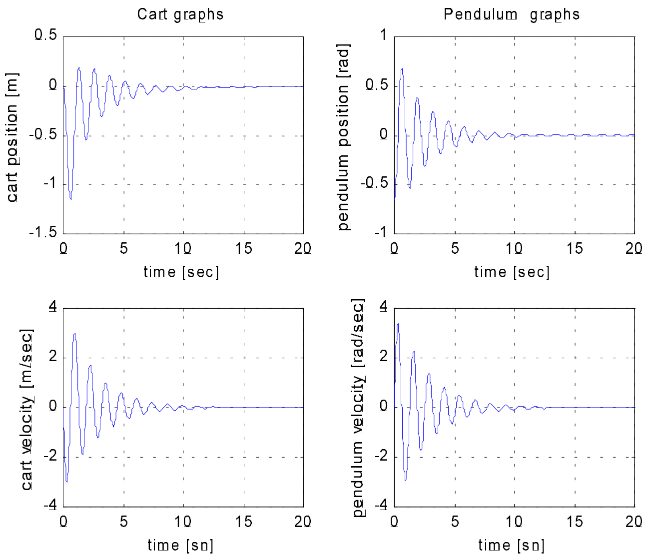

Figure 7. Linearized system characteristics for a given initial angle of $\theta = 37°$.

As it is seen from the graphs, although the system constraints are not violated, the linearized system creates too much oscillation and they indicate that the time required for the system to reach stability is greatly increased. For example, the time for the pendulum to be in the upright position is approximately 10 seconds which is quite undesirable. Furthermore, Figure 8 gives us the stability characteristics of the linearized system. Note that, due to the characteristics of the linearized system, this figure does not give us sufficient information, i.e., for example, the interval of [-0.8 0.8] which seems as the θ value is lying in the maximized region of attraction does not reflect the real values for pendulum. Because when the linearized system is simulated above 40° degree, the system becomes unstable. However, for the sake of comparison, the stability region of the LPV system is simulated with the same initial angle, and shown in Figure 9.

To observe the characteristics of both systems, Figure 9 will be more helpful. As seen in this figure, LPV system has much better results.

CONCLUSION

In this paper, analytical approaches are developed for asymptotically stable nonlinear systems. Linear parameter varying method in which an assigned parameter is used to convert the nonlinear system into a linear one

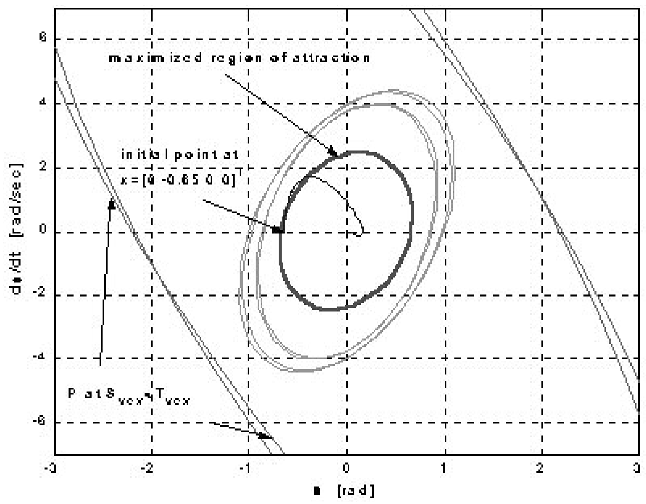

Figure 8. The stability region of the linearized system. An initial angle starting at $37°$ reaches to stability after oscillations.

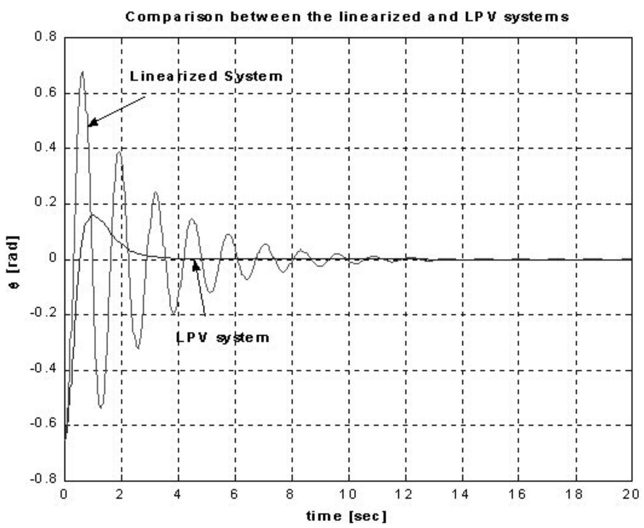

Figure 9. For comparison purposes, the LPV system is run for the same initial condition of $37°$ as in linearized form and the stability region is obtained.

is used for the control purposes. A region of attraction is formed by the constraints imposed on the system. Since the volume of the region of attraction and the system performance are inversely proportional, a maximized region of attraction is obtained under the protection of better performance. While these approaches are developed in association with a particular control system, the general analytic framework is applicable to other control applications without much difficulty.

The applied methods are the applications of Lyapunov stability theory and Linear Matrix Inequality (LMI) methodology to the development of an asymptotically stable controller. Namely, full state feedback and dynamic output feedback methods are used as the analysis and synthesis techniques. These techniques provide a mathematical basis for confirming the safety region of the controller and deriving the safety control laws.

Finally, a comprehensive application of LPV control techniques to the control of a cart and inverted pendulum system is presented. Undoubtedly, the major obstacles for stabilizing this system are the implementation constraints that put hard limitations on the controller dynamics. The difficulty of handling that problem with the currently known LPV techniques are eliminated by using the LMI techniques. These implementation constraints, known as physical constraints, determine the region of attraction of the system in which the system is known to be stable. Two cases are considered: 1) deriving the region of attraction for the given physical constraints; and 2) constructing the region of attraction such that the volume is maximized. Throughout the paper, both the physical constraints and the conditions that render the region of attraction as maximized are introduced to the system as LMI's. In the inverted pendulum system, if the control gain is obtained such that the region of attraction is too large, the corresponding controller would take a longer time to drive the physical system to a neighborhood of the equilibrium state. Therefore, in the actual design, a trade-off must be considered between the volume gained and performance lost.

The simulation results give us quite satisfactory results. In dynamic output feedback control, the system performances are better than the classical design criteria and the obtained region of attraction guarantees the system's stability. However, in the linearized system, stability region cannot guarantee the stability of the system. Although the constructed graphs show a stability region for θ between $-50°$ and $50°$, the system cannot be stabilized for the values of $|\theta| > 40°$. Therefore, it can be generalized that the system with LPV control guarantees the system stability for all values of state parameters chosen in the stability region and the stability region of the LPV system is larger than the stability region of the linearized system.

While the inverted pendulum is a prototype system, it certainly contains a lot of control issues. It must be emphasized that the analytic approaches developed to address these issues can be very well extended to other control applications, including large-scale control systems. On the other hand, of course, there are still some unsolved problems and they can be the possible subjects of the future researches.

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