Influence of Physical Parameters on System Eigenvalues and Its Use in Physical Model Reduction

Âli Yurdun Orbak
Industrial Engineering Department
Uludağ University
Bursa, Turkey 16059
Email: orbak@uludag.edu.tr

Eşref Eşkinat
Mechanical Engineering Department
Boğaziçi University
Istanbul, Turkey 34342
Email: eskinat@boun.edu.tr

Abstract—**The identification of subsystems and / or components that determine a given eigenvalue of the overall system is a challenging and important topic. This paper proposes a set of theorems and definitions that lead to an efficient procedure for this purpose. In the procedure, the bond graph representation of dynamic systems is utilized since they lead to better understanding of the system structure. The procedures include the calculation of eigenvectors. After this calculation an "effect" matrix is produced that indicates the relative importance of physical parameters in a selected eigenvalue. In addition to the relative importance, an efficient physical model reduction procedure can be constructed. Two examples are given to illustrate the approach.**

Index Terms—**Modelling, bond graphs, eigenvalue sensitivity, physical model reduction.**

I. Introduction

The importance of obtaining sensitivities for an eigenvalue problem stems from the fact that the partial derivatives of the system matrices' with respect to the system parameters are important for several analysis issues such as efficient design modifications and gaining insight into the reasons for discrepancies between structural analysis and dynamic tests.

In this context, eigenvalue derivatives are also useful for determining the sensitivity of dynamic responses to system parameter variations. For example, knowledge of the eigenvector derivatives with respect to physical parameters can help an engineer optimize a structural design or minimize its sensitivity to parameters. Such information can be used regularly for structural design and optimization, and for the improvement of the agreement between analytical and experimental results. Furthermore, for structural control systems, eigen derivatives can be directly applied to system identification and robust performance tests.

In the last two decades, several methods have been proposed to analyze the connection between a system variable and eigenvalues (modes) [3], [2], [19], [20]. For example, the participation factor approach has been extensively used for the analysis of power systems [18]. Participation factors are considered as a measure of the weights of the participation of modes in state components. Dynamic systems with large number of state variables, such as power systems, are often very complex. Thus, the physical knowledge of the system might help for analysis: knowing that the system presents an oscillatory behavior, the interest might be focused on a particular system eigenvalue (mode), by looking for the physical state variables most involved in the oscillation, without studying the entire system. In such cases, the participation factors might be useful in exploring the state variables that are relevant in the evolution of a particular eigenvalue (mode).

Since the participation factors can be used to detect the states that are most involved in an eigenvalue (mode), it is clear that once the eigenvalues of interest are identified, participation factors might help to obtain a reduced order model of the system which conserves the dynamics of interest. Although this is the case, the most important problem is still the identification of subsystems / components that determine a given eigenvalue.

In this paper, in order to give a reasonable solution to the problem of finding a connection between physical parameters of a system and its eigenvalues, the use of eigenvalue sensitivity using special state-space descriptions has been investigated. For this aim, a general analysis of participation factors and its relation to residues and eigenvalues is also performed. Throughout the research, it has been observed that the use of matrices with special components leads to clearer and simpler results. Analyzing the procedures of obtaining system matrices using bond graphs leads to a very efficient solution. Thus in the following sections, first the state-space representation discussed above will be briefly summarized. Then, the calculation of eigenvalue sensitivities using eigenvectors and their relationship to participation factors will be analyzed. Based on this analysis the "effect" matrix is introduced that indicates the relative importance of physical parameters in a selected eigenvalue.

II. State Space Representation of an LTI System

In this section, we will give a brief review of an existing procedure for the formulation of the state space equations of

0-7803-7729-X/03/$17.00 ©2003 IEEE.

LTI systems [1], [2]:

An LTI system can be characterized with several matrices that define the structure of the system. The parameters of the components can be described by two matrices, one for independent energy storage elements and one for dissipation elements [1]. The energy storage elements can be represented by the matrix $\mathbf{S}$, defined as

$$\mathbf{z} = \mathbf{S}\mathbf{x} \tag{1}$$

where x_i is the generalized momentum / displacement associated with the i'th independent energy storage element (state) and z_i is the flow / effort as the causal output of that element. For an LTI system with all of the independent energy storage elements (a total number of n) are one port, $\mathbf{S}$ is a diagonal matrix of the form $\text{diag}[s_1, s_2, \ldots, s_n]$, with s_i's as the parameters of the energy storage elements. From a computation point of view, if the i'th independent energy storage element is a capacitance (or an inertance), then $s_i = \frac{1}{C_i}$ (or respectively $s_i = \frac{1}{I_i}$).

On the other hand, the dissipation elements can be represented by the matrix $\mathbf{L}$, which contains the parameter values as follows: $\mathbf{d}_{out} = \mathbf{L}\mathbf{d}_{in}$, where d_{in_j} and d_{out_j} stand for the causal input and causal output of the j'th dissipation element, respectively. Again, for an LTI system with m one port dissipation elements, $\mathbf{L}$ is diagonal and of the form $\text{diag}[l_1, l_2, \ldots, l_m]$. From a computation point of view, when the j'th dissipation component has a flow (or an effort) as the causal input and an effort (respectively a flow) as the causal output, then $l_j = R_j$ (respectively $l_j = \frac{1}{R_j}$).

Then, the structure of a system is described by:

$$\dot{\mathbf{x}} = \mathbf{J}_{SS}\mathbf{z} + \mathbf{J}_{SL}\mathbf{d}_{out} + \mathbf{J}_{SU}\mathbf{u} \tag{2}$$

$$\mathbf{d}_{in} = \mathbf{J}_{LS}\mathbf{z} + \mathbf{J}_{LL}\mathbf{d}_{out} + \mathbf{J}_{LU}\mathbf{u} \tag{3}$$

where,

- $\mathbf{J}_{SS}$ describes the connections among the outputs of the energy storage elements and the inputs of the energy storage elements.
- $\mathbf{J}_{SL}$ describes the connections among the outputs of the dissipation elements and the inputs of the energy storage elements.
- $\mathbf{J}_{LS}$ describes the connections among the outputs of the energy storage elements and the inputs of the dissipation elements.
- $\mathbf{J}_{LL}$ describes the connections among the outputs of the dissipation elements and the inputs of the dissipation elements.
- $\mathbf{J}_{SU}$ describes the connections among the inputs from the sources $\mathbf{u}$ and the inputs of the energy storage elements.
- And $\mathbf{J}_{LU}$ describes the connections among the inputs from the sources $\mathbf{u}$ and the inputs of the dissipation elements.

As a result, the system's state space equation is given by $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$, where

$$\mathbf{A} = \left[\mathbf{J}_{SS} + \mathbf{J}_{SL}\mathbf{L}(\mathbf{I} - \mathbf{J}_{LL}\mathbf{L})^{-1}\mathbf{J}_{LS}\right]\mathbf{S} \equiv \mathbf{J}\mathbf{S} \tag{4}$$

$$\mathbf{B} = \mathbf{J}_{SU} + \mathbf{J}_{SL}\mathbf{L}(\mathbf{I} - \mathbf{J}_{LL}\mathbf{L})^{-1}\mathbf{J}_{LU} \tag{5}$$

In the next section the calculation of sensitivities of eigenvalues using eigenvectors will be given.

III. Calculation of Eigenvalue Sensitivity Using Eigenvectors

Consider the linear time-invariant continuous time system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{6}$$

where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{A} \in \mathbb{R}^{n \times n}$.

Then using modal decomposition the system dynamic matrix can be written as:

$$\begin{aligned}\mathbf{A} &= \mathbf{U}\mathbf{\Lambda}\mathbf{V} \\ &= \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_1 & \cdots & 0 \\ & & & \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \\ \vdots \\ \mathbf{v}_n^T \end{bmatrix}\end{aligned} \tag{7}$$

where $\mathbf{U}$ and $\mathbf{V}$ are the right and left eigenvector matrices respectively, and $\mathbf{\Lambda}$ is a diagonal matrix with eigenvalues being the diagonal elements. It should be noted that the eigenvectors $\mathbf{u}_i$ and $\mathbf{v}_i$ can always be chosen so that $\mathbf{u}_i^T\mathbf{v}_i = 1$, or similarly in matrix form, $\mathbf{U}\mathbf{V} = \mathbf{V}\mathbf{U} = \mathbf{I}$. This means that $\mathbf{V}$ is the inverse of $\mathbf{U}$. Here the $\mathbf{A}$ matrix is assumed to have only simple eigenvalues. In case of repeated eigenvalues, the diagonal matrix becomes a Jordan form matrix,and the corresponding right and left eigenvectors will become generalized eigenvectors. The use of generalized eigenvectors and Jordan form do not alter the derivations given in the next sections.

Using the modal decomposition with given initial conditions vector $\mathbf{x}(0)$, the solution of (6) can be described as:

$$\mathbf{x}(t) = \sum_{i=1}^{n} e^{\lambda_i t}\mathbf{u}_i\mathbf{v}_i^T\mathbf{x}(0) \tag{8}$$

From this equation, one can write the kth component of the state as follows:

$$\begin{aligned}\mathbf{x}^k(t) &= \sum_{i=1}^{n} e^{\lambda_i t}, \mathbf{u}_i^k\mathbf{v}_i^T\mathbf{x}(0) \\ &= \sum_{i=1}^{n} e^{\lambda_i t}\mathbf{u}_i^k \left[\mathbf{v}_i^k\mathbf{x}^k(0) + \sum_{j=1, j\neq i}^{n} \mathbf{v}_i^j\mathbf{x}^j(0)\right] \\ &= \sum_{i=1}^{n} e^{\lambda_i t}p_{ki}\mathbf{x}^k(0) + \sum_{i=1}^{n} e^{\lambda_i t}\left[\sum_{j=1, j\neq k}^{n} p_{kij}\mathbf{x}^j(0)\right]\end{aligned} \tag{9}$$

where

$$\begin{aligned} p_{ki} &\triangleq \mathbf{u}_i^k\mathbf{v}_i^k && \text{participation factor} \\ p_{kij} &\triangleq \mathbf{u}_i^k\mathbf{v}_i^j && \text{generalized participation} \end{aligned} \tag{10}$$

Here, participation factor, p_{ki}, can be understood as the weight of the participation of i-th mode in the k-th state component. Then a participation matrix can be formed as:

$$\Rightarrow \quad \mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} \tag{11}$$

For this participation matrix, $\mathbf{P}$, and generalized participation values, the following properties can be identified:

1) $\sum_{i=1}^{n} p_{ki} = 1$, i.e. rows of $\mathbf{P}$ sum to one.
2) $\sum_{k=1}^{n} p_{ki} = 1$, i.e. columns of $\mathbf{P}$ sum to one.
3) $\sum_{i=1} p_{kij} = 0$.

In addition to the above basic properties, the following theorem can be constructed:

Theorem 1: The generalized participation values are considered as the sensitivities of the eigenvalues of the matrix $\mathbf{A}$:

$$p_{kij} = \frac{\partial \lambda_i}{\partial a_{jk}} \tag{12}$$

where a_{jk} represents the jk-th element of matrix $\mathbf{A}$.

Proof:

$$\mathbf{v}_i^T \mathbf{A} \mathbf{u}_i = \lambda_i \mathbf{v}_i^T \mathbf{u}_i = \lambda_i$$

Then,

$$\begin{aligned}
\frac{\partial \lambda_i}{\partial q} &= \frac{\partial (\mathbf{v}_i^T \mathbf{A} \mathbf{u}_i)}{\partial q} \\
&= \frac{\partial \mathbf{v}_i^T}{\partial q} \underbrace{\mathbf{A}\mathbf{u}_i}_{\lambda_i \mathbf{u}_i} + \mathbf{v}_i^T \frac{\partial (\mathbf{A}\mathbf{u}_i)}{\partial q} \\
&= \lambda_i \frac{\partial \mathbf{v}_i^T}{\partial q} \mathbf{u}_i + \mathbf{v}_i^T \left(\frac{\partial \mathbf{A}}{\partial q} \mathbf{u}_i + \mathbf{A} \frac{\partial \mathbf{u}_i}{\partial q} \right) \\
&= \lambda_i \frac{\partial \mathbf{v}_i^T}{\partial q} \mathbf{u}_i + \mathbf{v}_i^T \frac{\partial \mathbf{A}}{\partial q} \mathbf{u}_i + \underbrace{\mathbf{v}_i^T \mathbf{A}}_{\lambda_i \mathbf{v}_i^T} \frac{\partial \mathbf{u}_i}{\partial q} \\
&= \mathbf{v}_i^T \frac{\partial \mathbf{A}}{\partial q} \mathbf{u}_i + \lambda_i \frac{\partial \mathbf{v}_i^T}{\partial q} \mathbf{u}_i + \lambda_i \mathbf{v}_i^T \frac{\partial \mathbf{u}_i}{\partial q} \\
&= \mathbf{v}_i^T \frac{\partial \mathbf{A}}{\partial q} \mathbf{u}_i + \underbrace{\lambda_i \frac{\partial (\mathbf{v}_i^T \mathbf{u}_i)}{\partial q}}_{= 0 \text{ as } \mathbf{v}_i^T \mathbf{u}_i = 1} \\
&= \mathbf{v}_i^T \frac{\partial \mathbf{A}}{\partial q} \mathbf{u}_i
\end{aligned}$$

If the parameter q is the element of a_{jk} of the matrix $\mathbf{A}$, then $\frac{\partial \mathbf{A}}{\partial a_{jk}}$ is a matrix whose elements are all zero and the element in j-th row and k-th column is one. Thus, one can write,

$$\frac{\partial \mathbf{A}}{\partial a_{jk}} = \mathbf{e}_j \mathbf{e}_k^T$$

where $\mathbf{e}_j$ and $\mathbf{e}_k$ are the j-th and k-th column of and identity matrix $\mathbf{I}_{n \times n}$, respectively. As a result,

$$\frac{\partial \lambda_i}{\partial a_{jk}} = \mathbf{v}_i^T \mathbf{e}_j \mathbf{e}_k^T \mathbf{u}_i = \mathbf{v}_i^j \mathbf{u}_i^k \equiv p_{kij}$$

□ End of proof.

It should be noted that this proof directly leads to the result that participation factors are the sensitivities of the diagonal terms of $\mathbf{A}$, i.e.:

$$p_{ki} = \frac{\partial \lambda_i}{\partial a_{kk}}$$

From the descriptions above the following theorem can also be deduced:

Theorem 2: The entries of the system matrix $\mathbf{A}$ can be expressed as a linear combination of the eigenvalues with the coefficients being the participation values.

Proof:

The $\mathbf{A}$ matrix can be written as:

$$\mathbf{A} = \sum_{i=1}^{n} \lambda_i \mathbf{u}_i \mathbf{v}_i^T$$

utilizing the dyadic form. Then,

$$\begin{aligned}
a_{kj} &= \mathbf{e}_k^T \mathbf{A} \mathbf{e}_j \\
&= \sum_{i=1}^{n} \lambda_i \underbrace{\mathbf{e}_k^T \mathbf{u}_i}_{\mathbf{u}_i^k} \overbrace{\mathbf{v}_i^T \mathbf{e}_j}^{\mathbf{v}_i^j} \\
&= \sum_{i=1}^{n} \lambda_i p_{kij}
\end{aligned}$$

Specifically, for the diagonal elements,

$$a_{kk} = \sum_{i=1}^{n} \lambda_i p_{ki}$$

is obtained.

□ End of proof.

Furthermore, the following lemma can be written without much difficulty.

An examination with the use of linear algebra gives the following simple connection between the participation values (p_{kij}) and partial fraction expansion residues (R_i):

Lemma 1: As one can write,

$$(s\mathbf{I} - \mathbf{A})^{-1} = \sum_{i=1}^{n} \frac{R_i}{s - \lambda_i}$$

and

$$\begin{aligned}
e^{\mathbf{A}t} &= e^{\mathbf{U}\mathbf{\Lambda}\mathbf{V}t} = \mathbf{U} e^{\mathbf{\Lambda}t} \mathbf{V} \\
&= \sum_{i=1}^{n} e^{\lambda_i t} \mathbf{u}_i \mathbf{v}_i^T \quad \Rightarrow \quad R_i = \mathbf{u}_i \mathbf{v}_i^T
\end{aligned}$$

then the participation values can be written as:

$$\begin{aligned}
p_{kij} &\triangleq \mathbf{u}_i^k \mathbf{v}_i^j \\
&= \mathbf{e}_k^T \underbrace{\mathbf{u}_i \mathbf{v}_i^T}_{R_i} \mathbf{e}_j \\
\Rightarrow p_{kij} &\triangleq \mathbf{e}_k^T R_i \mathbf{e}_j
\end{aligned}$$

The above definitions and theorems leads to a better understanding of the relationship between states and physical parameters. In what follows, by the use of the special form of the state-space equations obtained using bond graphs, the "effect" matrix will be introduced.

IV. Effect Matrix

In the previous sections the following has been derived:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{13}$$

where

$$\mathbf{A} = (\mathbf{J}_{SS} + \mathbf{J}_{SL}\mathbf{L}(\mathbf{I} - \mathbf{J}_{LL}\mathbf{L})^{-1}\mathbf{J}_{LS})\mathbf{S} \equiv \mathbf{J}\mathbf{S} \tag{14}$$

Using this special form of the state-space equations, the following can be derived from the theorems of the previous section:

$$\frac{\partial \lambda_i}{\partial (IC)_j} = \mathbf{v}_i^T (\mathbf{J} \frac{\partial \mathbf{S}}{\partial (IC)_j}) \mathbf{u}_i \tag{15}$$

where IC stands for energy storage elements. It should be noted that the partial derivative on the right hand side of this equations is simply $\mathbf{e}_j \mathbf{e}_j^T$ as the matrix $\mathbf{S}$ is diagonal. Further more for the energy dissipation elements the following can be derived:

$$\frac{\partial \lambda_i}{\partial r_j} = \mathbf{v}_i^T (\mathbf{J}_{SL} \frac{\partial \mathbf{L}}{\partial r_j} \mathbf{J}_{LS} \mathbf{S}) \mathbf{u}_i \tag{16}$$

where r stands for energy dissipation elements. Here, for simplicity, it is assumed that $\mathbf{J}_{LL} \equiv 0$, i.e. none of the dissipation elements are directly casually related. This assumption is not a critical assumptions as this is a common case in dynamic systems, especially in structures. It should be noted that when this assumption is valid $\mathbf{A}$ becomes $\mathbf{A} = (\mathbf{J}_{SS} + \mathbf{J}_{SL}\mathbf{L}\mathbf{J}_{LS})\,\mathbf{S}$. Similar to the energy storage case the partial derivative on the right hand side of this equations is simply $\mathbf{e}_j \mathbf{e}_j^T$ as the matrix $\mathbf{L}$ is diagonal.

Now that we have calculated the eigenvalue sensitivities we can form and define two "effect" matrices, namely, one for energy storage, $\mathbf{E}_{IC}$, and one for energy dissipation elements, $\mathbf{E}_R$.

In conclusion the following four step procedure can be employed to calculate the relative contribution of physical elements on a selected eigenvalue by forming two matrices:

1) After forming the bond graph of the system, calculate the matrices $\mathbf{S}$, $\mathbf{J}_{SS}$, $\mathbf{L}$, $\mathbf{J}_{SL}$, $\mathbf{J}_{LS}$, $\mathbf{J}_{LL}$ and $\mathbf{A}$.
2) Calculate the left and right eigenvector matrices $\mathbf{V}$ and $\mathbf{U}$.
3) For each eigenvalue calculate the sensitivities using equations (15) and (16).
4) Form all the sensitivity values for all eigenvalues in matrix form such that each row corresponds to one eigenvalue, and each column corresponds to one energy storage or energy dissipation element (Matrices $\mathbf{E}_{IC}$ and $\mathbf{E}_R$).

The resulting two effect matrices directly gives information on the sensitivity of all eigenvalues. Furthermore, one can directly see the effect of each physical element's effect on all eigenvalues.

It is important to note that the introduction of the effect matrix, $\mathbf{E}_{IC}$ constitutes the superset of a method developed in [2].

In addition to its above mentioned efficient use, the effect matrices lead to the physical model reduction of dynamic systems, i.e. the physical parameters that do not affect an eigenvalue of interest can be removed. Furthermore, physical elements can be put in subsystems that define their specific behavior. This can be accomplished by looking at the effect matrices and by checking their relationship using the bond graph causality assignment.

V. Examples

In this section two examples will be given, one for distinct eigenvalues and one for repeated eigenvalues. The first example is a physical one, namely the linearized hydraulic line of a power steering system. The second example is a standard bond graph system that can easily be found in electrical or mechanical systems.

A. Hydraulic Line of a Power Steering System

In this section, the eigenvalue sensitivity method will be applied to a linearized hydraulic line of a power steering system shown schematically in Figure 1. The names of the parameters on the bond graph are tabulated in Table I. Detailed calculations of these parameters can be found in [3]. The hydraulic line is assumed to be open to air at the valve end. The effective resistance of the rotary valve indicated by R_V will thus be zero ($R_V = 0$ Nsec/m^5). The parameter values for the pipes and the hoses are tabulated in Tables II and III. This hydraulic line is of order 6.

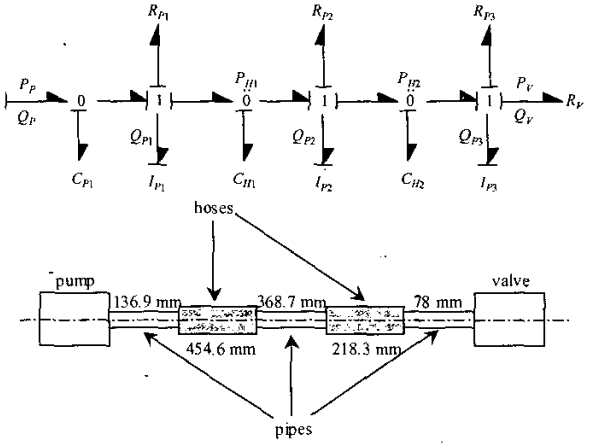

Fig. 1. Schematic and bond graph representation of a hydraulic line.

For this system, the following system matrices can be constructed:

$$\mathbf{S} = \begin{bmatrix} \frac{1}{I_{P1}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{I_{P2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{I_{P3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{C_{P1}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{C_{H1}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{C_{H2}} \end{bmatrix} \tag{17}$$

TABLE I

IMPORTANT VARIABLES IN BOND GRAPH REPRESENTATION OF THE HYDRAULIC LINE

Q_P	Actual pump flow rate to the outer system
P_P	Back pressure determined by the outer system (or pressure at the pump outlet port)
R_{Pn}	Resistance associated with the n'th pipe, $n = 1 \ldots 3$.
I_{Pn}	Inertance associated with the n'th pipe, $n = 1 \ldots 3$.
C_{Pn}	Capacitance associated with the n'th pipe, $n = 1 \ldots 3$.
C_{Hn}	Capacitance associated with the n'th hose, $n = 1 \ldots 2$.
Q_V	Flow rate from the hydraulic line

TABLE II

PARAMETER VALUES FOR PIPES

Pipe #	l_P [m]	I_P $[\frac{kg}{m^4}]$	R_P $[\frac{N\,sec}{m^5}]$	C_P $[\frac{m^5}{N}]$
1	1.369×10^{-1}	1.45×10^{6}	5.30×10^{6}	6.72×10^{-15}
2	3.687×10^{-1}	3.90×10^{6}	1.43×10^{7}	1.81×10^{-14}
3	7.8×10^{-2}	8.24×10^{5}	3.02×10^{6}	3.83×10^{-15}

$$\mathbf{J}_{SS} = \begin{bmatrix} 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \end{bmatrix} \tag{18}$$

$$\mathbf{L} = \begin{bmatrix} R_{P1} & 0 & 0 \\ 0 & R_{P2} & 0 \\ 0 & 0 & R_{P3} \end{bmatrix} \quad \mathbf{J}_{LL} = \mathbf{0}_{3\times 3} \tag{19}$$

$$\mathbf{J}_{SL} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{20}$$

$$\mathbf{J}_{LS} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{21}$$

With these matrices the $\mathbf{A}$ matrix of the system is obtained as:

$$\mathbf{A} = \begin{bmatrix} \frac{-R_{P1}}{I_{P1}} & 0 & 0 & \frac{1}{C_{P1}} & \frac{-1}{C_{H1}} & 0 \\ 0 & \frac{-R_{P2}}{I_{P2}} & 0 & 0 & \frac{1}{C_{H1}} & \frac{-1}{C_{H2}} \\ 0 & 0 & \frac{-R_{P3}}{I_{P3}} & 0 & 0 & \frac{1}{C_{H2}} \\ \frac{-1}{I_{P1}} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{I_{P1}} & \frac{-1}{I_{P2}} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{I_{P2}} & \frac{-1}{I_{P3}} & 0 & 0 & 0 \end{bmatrix} \tag{22}$$

TABLE III

PARAMETER VALUES FOR HOSES

Hose #	l_H [m]	C_H $[\frac{m^5}{N}]$
1	4.546×10^{-1}	1.67×10^{-12}
2	2.183×10^{-1}	8.0×10^{-13}

As a result, using Matlab, the eigenvalues of this $\mathbf{A}$ matrix are computed as:

$$\begin{aligned} \lambda_{1,2} &= -1.8272 \pm 10151i \\ \lambda_{3,4} &= -1.8327 \pm 1366.1i \\ \lambda_{5,6} &= -1.8332 \pm 352.74i \end{aligned}$$

After the calculation of associated eigenvectors of these eigenvalues, the effect matrices are calculated as:

$$\mathbf{E}_{IC} = 1.0e{+}017 * \begin{bmatrix} 0 & 0 & 0 & 7.5223 & 0.0001 & 0 \\ 0 & 0 & 0 & 7.5223 & 0.0001 & 0 \\ 0 & 0 & 0 & 0.0001 & 0.0001 & 0.0084 \\ 0 & 0 & 0 & 0.0001 & 0.0001 & 0.0084 \\ 0 & 0 & 0 & 0.0010 & 0.0010 & 0 \\ 0 & 0 & 0 & 0.0010 & 0.0010 & 0 \end{bmatrix}$$

$$\mathbf{E}_R = 1.0e-006 * \begin{bmatrix} 0.3448 & 0 & 0 \\ 0.3448 & 0 & 0 \\ 0 & 0.0257 & 0.4852 \\ 0 & 0.0257 & 0.4852 \\ 0 & 0.1025 & 0.1216 \\ 0 & 0.1025 & 0.1216 \end{bmatrix}$$

Here, in matrix $\mathbf{E}_{IC}$ each column corresponds to one energy storage element (in the order of matrix $\mathbf{S}$), and each row correspond to one eigenvalue. Similarly, in matrix $\mathbf{E}_R$ each column corresponds to one energy dissipation element (in the order of matrix $\mathbf{L}$), and each row correspond to one eigenvalue. From the effect matrices, one can observe that R_{P1} and I_{P1} have the most effect in $\lambda_{1,2}$. This result is consistent with the previous result [3] that the pipe resistance, R_{P1}, needs to be increased in order to eliminate the high vibration value. This parameter is directly related to the length, so we should change the length to get a reduced vibration frequency.

B. A Simple Physical Example with Repeated Roots

Consider the system given by its bond graph in Figure 2, [2]. All parameter values except α are shown on the figure. For this example $\alpha = 1$ is chosen.

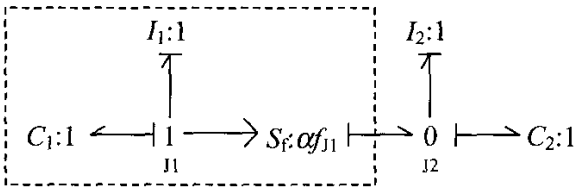

Fig. 2. A simple physical system.

For this system, using the same approach as in the first example the following system matrices can be constructed:

$$\mathbf{S} = \begin{bmatrix} \frac{1}{I_1} & 0 & 0 & 0 \\ 0 & \frac{1}{C_1} & 0 & 0 \\ 0 & 0 & \frac{1}{I_2} & 0 \\ 0 & 0 & 0 & \frac{1}{C_2} \end{bmatrix} \quad \mathbf{J} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \tag{23}$$

With these two matrices the **A** matrix of the system is obtained as:

$$\mathbf{A} = \mathbf{JS} = \begin{bmatrix} 0 & -\frac{1}{C_1} & 0 & 0 \\ \frac{1}{I_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{C_2} \\ \frac{1}{I_1} & 0 & -\frac{1}{I_2} & 0 \end{bmatrix} \quad (24)$$

This system produces the symbolic eigenvalues as: $\pm\sqrt{\frac{1}{I_1 C_1}}i$, and $\pm\sqrt{\frac{1}{I_2 C_2}}i$. With the chosen parameter values the numerical eigenvalues are calculated as: $\pm i$, $\pm i$, which indicates that we have two repeated roots. Thus we will have generalized eigenvectors. As a result, using Matlab the right and left eigenvector matrices for this **A** matrix are computed as:

$$\mathbf{U} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4 \end{bmatrix} \quad (25)$$

$$\mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{bmatrix} \quad (26)$$

where

$$\mathbf{u}_1 = \begin{bmatrix} 0 \\ 0 \\ -0.2500i \\ 0.2500 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0.5000 \\ -0.5000i \\ 0 \\ -0.2500i \end{bmatrix}$$

$$\mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 0.2500i \\ 0.2500 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} 0.5000 \\ 0.5000i \\ 0 \\ 0.2500i \end{bmatrix}$$

and

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ -1.0000 \\ -2.0000i \\ 2.0000 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1.0000 \\ -1.0000i \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{v}_3 = \begin{bmatrix} 0 \\ 1.0000 \\ 2.0000i \\ 2.0000 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 1.0000 \\ 1.0000i \\ 0 \\ 0 \end{bmatrix}$$

As this system has repeated eigenvalues the eigenvectors are the generalized eigenvectors. But as it has been explained before, this does not alter the result. Thus the effect matrix, $\mathbf{E}_{IC}$ (there are no dissipation elements in this system), is calculated as:

$$\mathbf{E}_{IC} = \begin{bmatrix} 0 & 0 & 0.5000 & 0.5000 \\ 0.5000 & 0.5000 & 0 & 0 \\ 0 & 0 & 0.5000 & 0.5000 \\ 0.5000 & 0.5000 & 0 & 0 \end{bmatrix}$$

In this matrix each column corresponds to one energy storage element (in the order of matrix **S**), and each row correspond to one eigenvalue, specifically, in this case the eigenvalues are in order of $i, i, -i, -i$. It can be observed that the results are remarkable. The effect matrix directly indicates that the weights of physical components on the eigenvalues, for this set of parameters, are the same. It can also be observed that only $I_1 - C_1$ effect one set of eigenvalues, and $I_2 - C_2$ effect the other. This is consistent with the symbolic calculation obtained.

VI. Conclusions

In this paper, a set of theorems and definitions that lead to an efficient procedure for the identification of subsystems and / or components that determine a given eigenvalue of the overall system is proposed. In the procedure, a special type of state-space description obtained from bond graphs is utilized. After the calculation of eigenvectors and the defined "effect" matrices, the relative importance of physical parameters in a selected eigenvalue is readily obtained. Two examples are given to illustrate the results.

References

[1] Rosenberg, R., "State-Space Formulation for Bond Graph Models of Multiport Systems.", *Journal of Dynamic Systems, Measurement and Control*, March 1971.

[2] Ye, Y. and K. Youcef-Toumi, "Subsystem's Influence on a System Eigenvalue", in *Proceedings of the IEEE Southeastcon 2000*, pp. 261–267, IEEE, April 2000.

[3] Orbak, A. Y., K. Youcef-Toumi, and Masaaki Senga, "Model Reduction and Design of a Power Steering System", in *Proceedings of the American Control Conference*, pp. 4476–4481, June 1999.

[4] Paynter, H. M., *Analysis and Design of Engineering Systems*, MIT Press, Cambridge, MA, 1961.

[5] Karnopp D. C. and R. C. Rosenberg, *Analysis and Simulation of Multiport Systems - The Bond Graph Approach to Physical System Dynamics*, MIT Press, Cambridge, MA, 1968.

[6] Karnopp D. C. and R. C. Rosenberg, *System Dynamics, A Unified Approach*, John Wiley, New York, NY, 1975.

[7] Karnopp D. C., D. L. Margolis, and R. C. Rosenberg, *System Dynamics, A Unified Approach*, John Wiley, New York, NY, 2nd ed., 1990.

[8] Rosenberg, R. C. and D. C. Karnopp, *Introduction to Physical System Dynamics*, McGraw Hill, New York, NY, 1983.

[9] Thoma, J. U., *Introduction to Bond Graphs and Their Applications*, Pergamon Press, Oxford, 1975.

[10] Van Dixhoorn, J. J., "Bond Graphs and the Challenge of a Unified Modelling Theory of Physical Systems", in F. E. Cellier (Ed.), *Progress in Modelling and Simulation*, pp. 207–245, Academic Press, New York, 1982.

[11] Breedveld, P. C., *Physical Systems Theory in Terms of Bond Graphs*, Ph.D. Thesis, University of Twente, Enschede, Netherlands, 1984.

[12] Breedveld, P. C., "Multi-Bond Graph Elements in Physical Systems Theory", *Journal of the Franklin Institute*, Vol. 319, No. 1/2, pp. 1–36, 1985.

[13] Rosenberg, R. C., *A Users Guide to ENPORT-4*, Wiley, New York, 1974.

[14] Lorenz, F., *Modelling System 1, User's Manual*, Lorenz Simulation, Liege, Belgium, 1997.

[15] Granda, J. J., "Computer Generation of Physical System Differential Equations Using Bond Graphs", *Journal of the Franklin Institute*, Vol. 319, No. 1/2, pp. 243–256, 1985.

[16] Broenink, J. F., *Computer-Aided Physical-Systems Modeling and Simulation: A Bond-Graph Approach*, Ph.D. Thesis, University of Twente, Enschede, Netherlands, 1990.

[17] Broenink, J. F., "Modelling, Simulation and Analysis with 20-SIM", *Journal A, Special issue on CACSD*, Vol. 38, No. 3, pp. 22–25, 1997.

[18] Wilfert, H.-H., Voigtländer, K., and Erlich, I, "Dynamic Coherency Identification of Generators Using Self-organising Feature Maps", *Control Engineering Practice*, No. 9, pp. 769–775, 2001.

[19] Shaw, J. and Jayasuriya, S, "Modal Sensitivities for Repeated Eigenvalues and Eigenvalue Derivatives", *AIAA Journal*, Vol. 30, No. 3, pp. 850–852, 1992.

[20] Zimoch, R. Z., "Algorithm for Modification of Parameters in Vibrating Systems", *AIAA Journal*, Vol. 29, No. 9, pp. 1525–1527, 1991.